LESSON 14: THE STREETER-PHELPS MODEL

THE STREETER-PHELPS EQUATION

It is the first model in order to study the presence of Biodegradable Organic Material into a watercourse and its effects on the DO which is reduced cause of the Biological Oxygen Demand.

Let's define:
- B BOD concentration $[kg/m^3]$
- O DO concentration $[kg/m^3]$
- O_s Saturate concetration of DO $[kg/m^3]$
- $\tilde{O} = O_s - O$ Deficit of DO $[kg/m^3]$
- r_d Deoxygenation rate $[s^{-1}]$
- r_a Aeration rate $[s^{-1}]$

The mass conservation and the reaction process of BOD give the following equation system. The work hypotheses are uniform flow and negligible dispersion process.

$$\begin{cases} \dfrac{\partial B}{\partial t} + U\dfrac{\partial B}{\partial x} = -r_d B \\ \dfrac{\partial O}{\partial t} + U\dfrac{\partial O}{\partial x} = -r_d B + r_a \tilde{O} \\ \dfrac{\partial \tilde{O}}{\partial t} + U\dfrac{\partial \tilde{O}}{\partial x} = r_d B - r_a \tilde{O} \end{cases}$$

→ *BOD reduction.*

→ *DO evolution considering O_2 decay and production due to BOD redox and Aeration.*

→ *Deficit of DO evolution . It is complementary to DO.*

THE STREETER-PHELPS EQUATION

The analytical solution is easily determined by assuming stationarity, i.e.:

$$\begin{cases} U\dfrac{\mathrm{d}B}{\mathrm{d}x} = -r_d B & (1) \\ U\dfrac{\mathrm{d}O}{\mathrm{d}x} = -r_d B + r_a\tilde{O} & (2) \\ U\dfrac{\mathrm{d}\tilde{O}}{\mathrm{d}x} = r_d B - r_a\tilde{O} & (3) \end{cases}$$

The solution of (1) is: $B = B_0 e^{-r_d \frac{x}{U}}$

Whereas, focusing on (3) the equation can be rearranged as following:

$$\begin{cases} \dfrac{\mathrm{d}\tilde{O}}{\mathrm{d}x} + p\tilde{O} = q \\ \tilde{O}(0,0) = \tilde{O}_0 \end{cases}$$

In this case

$$p = r_a/U$$

$$q = \frac{r_d B}{U} = \frac{r_d B_0}{U} e^{-r_d \frac{x}{U}}$$

The solution of the system holds:

$$\tilde{O} = \tilde{O}_0 e^{-\gamma} + e^{-\gamma}\int_0^x e^{\gamma} q \,\mathrm{d}\xi$$

with

$$\gamma = \int_0^x p \,\mathrm{d}\xi = \frac{r_a x}{U}$$

THE STREETER-PHELPS EQUATION

By replacing the definition of γ and q, the equation reads:

$$\tilde{O} = \tilde{O}_0 e^{-\frac{r_a x}{U}} + e^{-\frac{r_a x}{U}} \int_0^x \frac{r_d B_0}{U} e^{-r_d \frac{\xi}{U}} e^{r_a \frac{\xi}{U}} \, \mathrm{d}\xi$$

$$\tilde{O} = \tilde{O}_0 e^{-\frac{r_a x}{U}} + \frac{r_d B_0}{U} e^{-\frac{r_a x}{U}} \int_0^x e^{(r_a - r_d)\frac{\xi}{U}} \, \mathrm{d}\xi$$

$$\int_a^b e^{f(x)} \, \mathrm{d}x = \frac{\left[e^{f(x)}\right]_a^b}{f'(x)}$$

$$\tilde{O} = \tilde{O}_0 e^{-\frac{r_a x}{U}} + \frac{r_d B_0}{\cancel{U}} e^{-\frac{r_a x}{U}} \frac{\cancel{U}}{r_a - r_d} \left[e^{(r_a - r_d)\frac{\xi}{U}}\right]_0^x$$

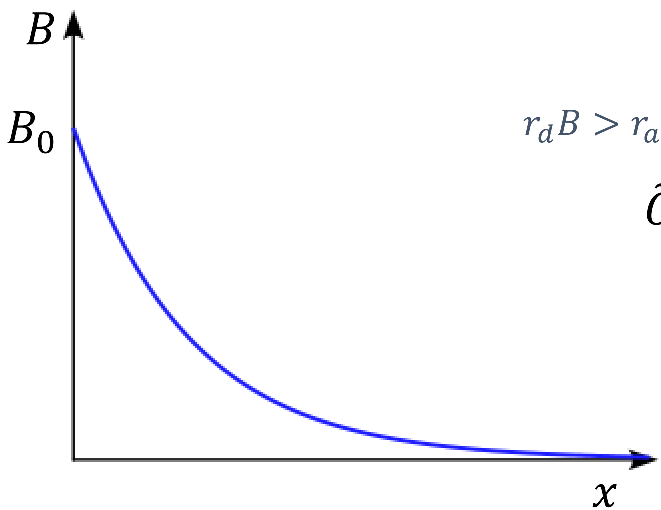
$$\tilde{O} = \tilde{O}_0 e^{-\frac{r_a x}{U}} + \frac{r_d B_0}{r_a - r_d} e^{-\frac{r_a x}{U}} \left[e^{(r_a - r_d)\frac{x}{U}} - 1\right]$$

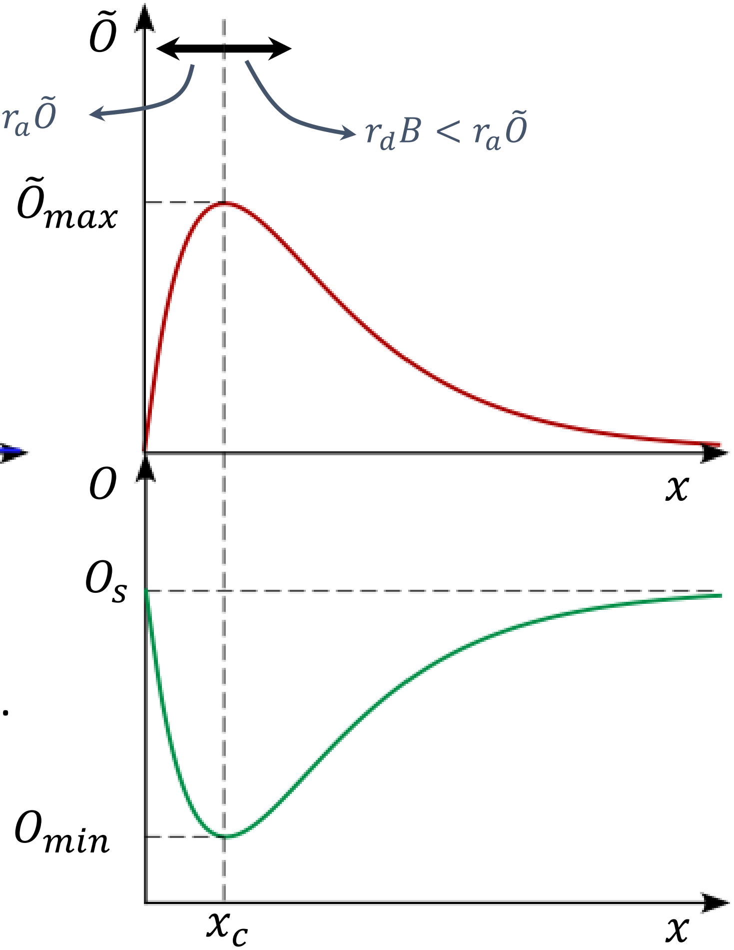
$$\longrightarrow \quad \tilde{O} = \tilde{O}_0 e^{-r_a \frac{x}{U}} + B_0 \frac{r_d}{r_a - r_d} \left[e^{-r_d \frac{x}{U}} - e^{-r_a \frac{x}{U}}\right]$$

When $\tilde{O}_0 = 0$, it is simplified in: $\tilde{O} = B_0 \frac{r_d}{r_a - r_d} \left[e^{-r_d \frac{x}{U}} - e^{-r_a \frac{x}{U}}\right]$

The DO is then:

$$O = O_s - \tilde{O} = O_s - \underbrace{(O_s - O_0)}_{\tilde{O}_0} e^{-r_a \frac{x}{U}} - B_0 \frac{r_d}{r_a - r_d}\left[e^{-r_d \frac{x}{U}} - e^{-r_a \frac{x}{U}}\right]$$





x_c is the Critic Distance, where O_{min} (and $\tilde{O}_{max}$). In this section: $r_d B = r_a \tilde{O}$.

N.B. The initial condition is $\tilde{O}_0 = 0$.

THE CRITIC CONDITION

Let's analyze the DO in x_c. $\tilde{O}_{max}$ (or O_{min}) means $\mathrm{d}\tilde{O}/\mathrm{d}x = 0$ (or $\mathrm{d}O/\mathrm{d}x = 0$), i.e.:

$$\frac{\mathrm{d}\tilde{O}}{\mathrm{d}x} + p\tilde{O} = q \quad \xrightarrow{\mathrm{d}\tilde{O}/\mathrm{d}x=0} \quad p\tilde{O} = q$$

That is: $$\frac{r_a \tilde{O}_c}{U} = \frac{r_d B_0}{U} e^{-r_d \frac{x_c}{U}} \longrightarrow \tilde{O}_c = \frac{r_d}{r_a} B_0 e^{-r_d \frac{x_c}{U}}$$

We can determine x_c by replacing $\tilde{O}_c$ into the analytical solution of $\tilde{O}$ in $x = x_c$. We find:

$$\frac{r_d}{r_a} B_0 e^{-r_d \frac{x_c}{U}} = \frac{\tilde{O}_0}{B_0} e^{-r_a \frac{x_c}{U}} + B_0 \frac{r_d}{r_a - r_d}\left[e^{-r_d \frac{x_c}{U}} - e^{-r_a \frac{x_c}{U}}\right]$$

$$\left(\frac{r_d}{r_a} - \frac{r_d}{r_a - r_d}\right) e^{-r_d \frac{x_c}{U}} = \left(\frac{\tilde{O}_0}{B_0} - \frac{r_d}{r_a - r_d}\right) e^{-r_a \frac{x_c}{U}}$$

$$\left[\frac{r_d r_a - r_d^2 - r_d r_a}{r_a (r_a - r_d)}\right] e^{-r_d \frac{x_c}{U}} = \left[\frac{(r_a - r_d)\tilde{O}_0 - r_d B_0}{B_0 (r_a - r_d)}\right] e^{-r_a \frac{x_c}{U}}$$

THE CRITIC CONDITION

By grouping the terms with the exponent the equation holds:

$$e^{(r_a-r_d)\frac{x_c}{U}} = -\left[\frac{r_a(r_a-r_d)}{r_d^{\,2}}\right]\left[\frac{(r_a-r_d)\tilde{O}_0 - r_d B_0}{B_0(r_a-r_d)}\right] = \frac{r_a r_d B_0 - r_a(r_a-r_d)\tilde{O}_0}{r_d^{\,2} B_0}$$

$$e^{(r_a-r_d)\frac{x_c}{U}} = \frac{r_a}{r_d} - \frac{r_a}{r_d}\frac{r_a-r_d}{r_d}\frac{\tilde{O}_0}{B_0} = \frac{r_a}{r_d}\left(1-\frac{r_a-r_d}{r_d}\frac{\tilde{O}_0}{B_0}\right)$$

ln

$$(r_a-r_d)\frac{x_c}{U} = \ln\left[\frac{r_a}{r_d}\left(1-\frac{r_a-r_d}{r_d}\frac{\tilde{O}_0}{B_0}\right)\right]$$

$$\longrightarrow \quad x_c = \frac{U}{r_a-r_d}\ln\left[\frac{r_a}{r_d}\left(1-\frac{r_a-r_d}{r_d}\frac{\tilde{O}_0}{B_0}\right)\right]$$

When initially DO is assumed being in saturated condition, i.e. $\tilde{O}_0 = 0$:

$$\longrightarrow \quad x_c = \frac{U}{r_a-r_d}\ln\left(\frac{r_a}{r_d}\right)$$

For the Streeter-Phelps model the critic codition is equal to:

$$x_c = \frac{U}{r_a - r_d} \ln\left[\frac{r_a}{r_d}\left(1 - \frac{r_a - r_d}{r_d}\frac{\tilde{O}_0}{B_0}\right)\right] \qquad \tilde{O}_c = \frac{r_d}{r_a} B_0 e^{-r_d \frac{x_c}{U}}$$

It is worth noting that:
- $x_c \propto r_a/r_d$ → <u>Purification Coefficient</u>
- $\tilde{O}_c \propto (r_a/r_d)^{-1}$
- $\tilde{O}_c \propto B_0$

Generally, The value of x_c and $\tilde{O}_c$ depends on other parameters. In particular:

i. Temperature T

ii. Flow rate Q

iii. Initial Deficit of Oxygen $\tilde{O}_0$

ROLE OF TEMPERATURE

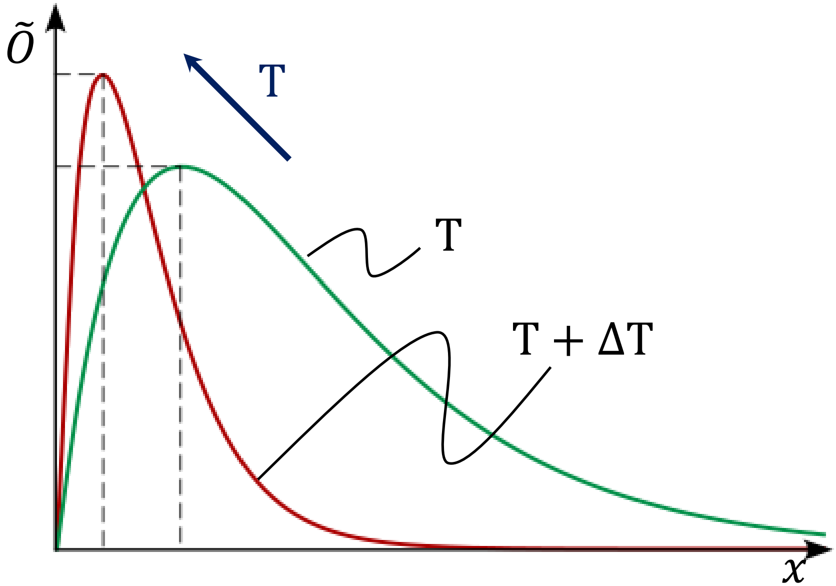
As we have seen: $T \uparrow \longrightarrow r \uparrow$

In this case:

$$\begin{cases} r_a = r_{a,20} 1.024^{T-20} \\ r_d = r_{d,20} 1.048^{T-20} \end{cases} \longrightarrow \frac{r_a}{r_d} \downarrow \quad \text{when T} \uparrow$$

Moreover the amount of oxygen to saturate water is smaller when T increasing, i.e.:

$$T \uparrow \longrightarrow O_s \downarrow$$



The overall effect is:

$$T \uparrow \longrightarrow \begin{matrix} \tilde{O}_c \uparrow \\ x_c \downarrow \end{matrix}$$

When Q increases:

- Initial BOD decreases ($B_0 \downarrow$) because the mass of BOD is constant, but the mass of water is greater.

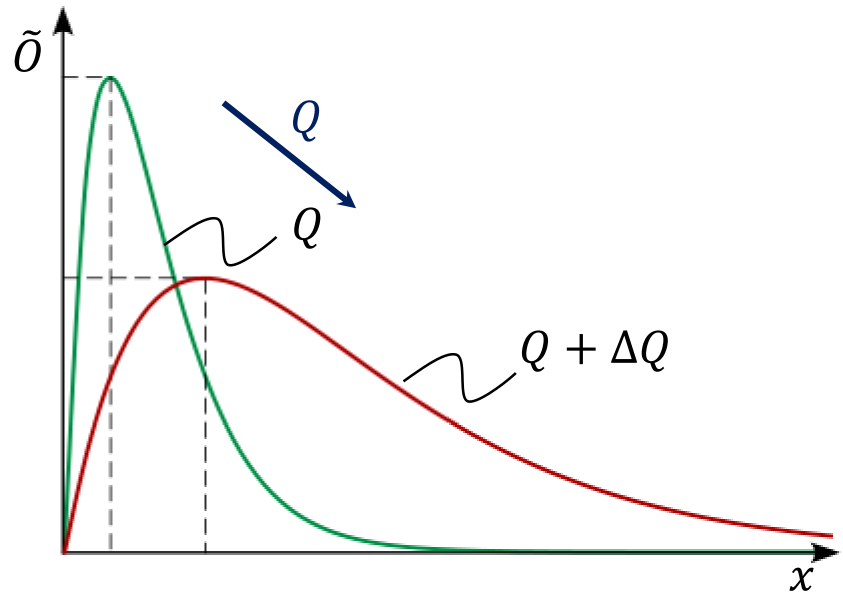
- Aeration r_a is affected in two counterposed ways:
  - $Q \uparrow \longrightarrow U \uparrow$ Higher U promotes the entrapment of air in the water
  - $Q \uparrow \longrightarrow z_0 \uparrow$ Higher z_0 means lower aeration because the exchange of air occurs only in the outer layer of water

$$\longrightarrow \quad r_a \propto \frac{U^m}{{z_0}^n} \quad m < 1, n > 1$$

The overall effect is: $Q \uparrow \longrightarrow r_a \downarrow$

By summarizing:

$$Q \uparrow \longrightarrow \begin{matrix} \tilde{O}_c \downarrow \\ x_c \uparrow \end{matrix}$$



ROLE OF INITIAL DEFICIT OF OXYGEN

When $\tilde{O}_0$ increases:

- $\tilde{O}_0 \uparrow \quad \longrightarrow \quad x_c \downarrow$ By the expression of x_c
- $\tilde{O}_0 \uparrow \quad \longrightarrow \quad \tilde{O}_c \uparrow$ Because the critic deficit of oxygen is inversely related by the critic distance x_c