ICEA

UNIVERSITÀ DEGLI STUDI DI PADOVA

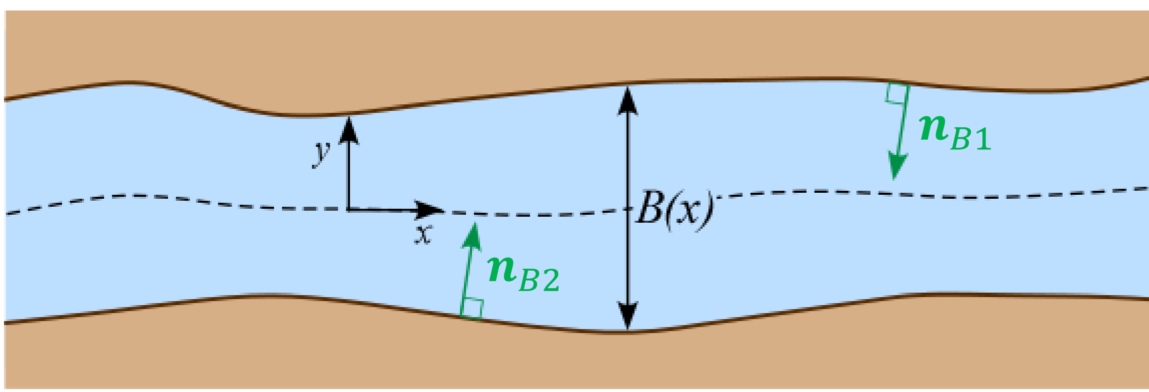
LESSON 11: 1D DISPERSION EQUATION AND DISPERSION COEFFICIENT

1D MASS BALANCE EQUATION

1D solution is due to the integration of 2D equation along y. In this case the quantities are averaged over the whole section.
It means that the diffusive process has to be fully developed along both the vertical and transverse dirction (section 3-3' of the skectch shown two lessons ago), i.e. $x > 100 \div 300\ B$ from insertion point.

The starting equation is:

$$\frac{\partial}{\partial t}(z_0 C) + \frac{\partial}{\partial x}(z_0 U_x C) + \frac{\partial}{\partial y}(z_0 U_y C) = \frac{\partial}{\partial x}\left[z_0 k_x \frac{\partial C}{\partial x}\right] + \frac{\partial}{\partial y}\left[z_0 k_y \frac{\partial C}{\partial y}\right]$$



$$F_{B1} = y - \frac{B(x,t)}{2} = 0$$

$$F_{B2} = -y + \frac{B(x,t)}{2} = 0$$

Considering the levees to be depended only on x: $F_{B1} = F_{B2} = y - \frac{1}{2}B(x)$

The kinematic conditions are:

$$\frac{\mathrm{d}F_B}{\mathrm{d}t} = 0 \quad \rightarrow \quad \left[\frac{1}{2}\frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial B}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t}\right]_{y=\pm B/2} = \left[\frac{1}{2}\frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial B}{\partial x}U_x - U_y\right]_{y=\pm B/2} = 0$$

The dynamic conditions are given by: $[\boldsymbol{q}^k \cdot \boldsymbol{n}_B]_{y=\pm B/2} = 0$

Where $\boldsymbol{q}^k$ is the flux due to dispersion, that is defined as: $\boldsymbol{q}^k = z_0\left(k_x\frac{\partial C}{\partial x}, k_y\frac{\partial C}{\partial y}\right)$

Then:

$$z_0\left[-\frac{k_x}{2}\frac{\partial C}{\partial x}\frac{\partial B}{\partial x} + k_y\frac{\partial C}{\partial y}\right]_{y=\pm B/2} = 0 \quad \longrightarrow \quad \left[-\frac{k_x}{2}\frac{\partial C}{\partial x}\frac{\partial B}{\partial x} + k_y\frac{\partial C}{\partial y}\right]_{y=\pm B/2} = 0$$

Similarly to the 2D case, we integrate between $-B/2$ and $B/2$:

$$\int_{-B/2}^{B/2}\frac{\partial}{\partial t}(z_0 C)\,\mathrm{d}y + \int_{-B/2}^{B/2}\frac{\partial}{\partial x}(z_0 U_x C)\,\mathrm{d}y + \int_{-B/2}^{B/2}\frac{\partial}{\partial y}(z_0 U_y C)\mathrm{d}y$$
$$= \int_{-B/2}^{B/2}\frac{\partial}{\partial x}\left[z_0 k_x\frac{\partial C}{\partial x}\right]\mathrm{d}y + \int_{-B/2}^{B/2}\frac{\partial}{\partial y}\left[z_0 k_y\frac{\partial C}{\partial y}\right]\mathrm{d}y$$

1D MASS BALANCE EQUATION

Keeping in mind the Leibniz rule of integration: $\int_\eta^H \frac{\partial f}{\partial x}\,\mathrm{d}z = \frac{\partial}{\partial x}\int_\eta^H f\,\mathrm{d}z - f\frac{\partial H}{\partial x}\Big|_{z=H} + f\frac{\partial \eta}{\partial x}\Big|_{z=\eta}$

$$\frac{\partial}{\partial t}\int_{-B/2}^{B/2} z_0 C\,\mathrm{d}y - \frac{1}{2}z_0 C\frac{\partial B}{\partial t}\Big|_{y=\frac{B}{2}} + \frac{1}{2}z_0 C\frac{\partial B}{\partial t}\Big|_{y=-\frac{B}{2}} + \frac{\partial}{\partial x}\int_{-B/2}^{B/2} z_0 U_x C\,\mathrm{d}y - \frac{1}{2}z_0 U_x C\frac{\partial B}{\partial x}\Big|_{y=\frac{B}{2}}$$
$$+\frac{1}{2}z_0 U_x C\frac{\partial B}{\partial x}\Big|_{y=-\frac{B}{2}} + z_0 U_y C\Big|_{y=\frac{B}{2}} - z_0 U_y C\Big|_{y=\frac{B}{2}} = \frac{\partial}{\partial x}\int_{-B/2}^{B/2} z_0 k_x\frac{\partial C}{\partial x}\,\mathrm{d}y - \frac{1}{2}z_0 k_x\frac{\partial C}{\partial x}\frac{\partial B}{\partial x}\Big|_{y=\frac{B}{2}}$$
$$+\frac{1}{2}z_0 k_x\frac{\partial C}{\partial x}\frac{\partial B}{\partial x}\Big|_{y=-\frac{B}{2}} + z_0 k_y\frac{\partial C}{\partial y}\Big|_{y=\frac{B}{2}} - z_0 k_y\frac{\partial C}{\partial y}\Big|_{y=-\frac{B}{2}}$$

By grouping:

$$\frac{\partial}{\partial t}\int_{-B/2}^{B/2} z_0 C\,\mathrm{d}y + \frac{\partial}{\partial x}\int_{-B/2}^{B/2} z_0 U_x C\,\mathrm{d}y - z_0 C\cancelto{0}{\left[\frac{1}{2}\frac{\partial B}{\partial t} + U_x\frac{1}{2}\frac{\partial B}{\partial x} - U_y\right]_{y=\frac{B}{2}}}$$
$$+ z_0 C\cancelto{0}{\left[\frac{1}{2}\frac{\partial B}{\partial t} + U_x\frac{1}{2}\frac{\partial B}{\partial x} - U_y\right]_{y=-\frac{B}{2}}} = \frac{\partial}{\partial x}\int_{-B/2}^{B/2} z_0 k_x\frac{\partial C}{\partial x}\,\mathrm{d}y + z_0\cancelto{0}{\left[-\frac{1}{2}k_x\frac{\partial C}{\partial x}\frac{\partial B}{\partial x} + k_y\frac{\partial C}{\partial y}\right]_{y=\frac{B}{2}}}$$
$$-z_0\cancelto{0}{\left[-\frac{1}{2}k_x\frac{\partial C}{\partial x}\frac{\partial B}{\partial x} + k_y\frac{\partial C}{\partial y}\right]_{y=-\frac{B}{2}}}$$

1D DISPERSION EQUATION

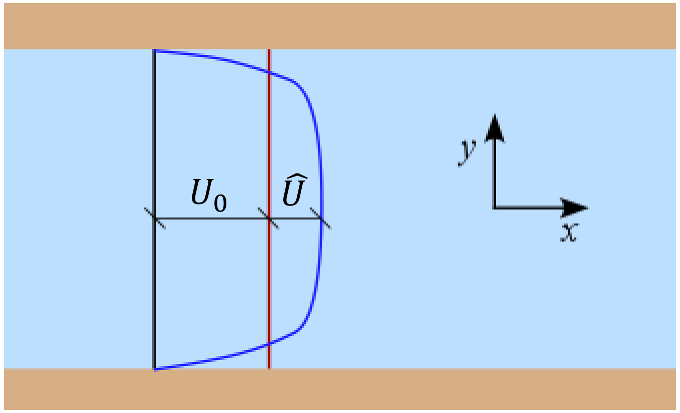
The integration of the 2D dispersion equation along y yields:

$$\frac{\partial}{\partial t}\int_{-B/2}^{B/2} z_0 C \, \mathrm{d}y + \frac{\partial}{\partial x}\int_{-B/2}^{B/2} z_0 U_x C \, \mathrm{d}y = \frac{\partial}{\partial x}\int_{-B/2}^{B/2} z_0 k_x \frac{\partial C}{\partial x} \mathrm{d}y$$

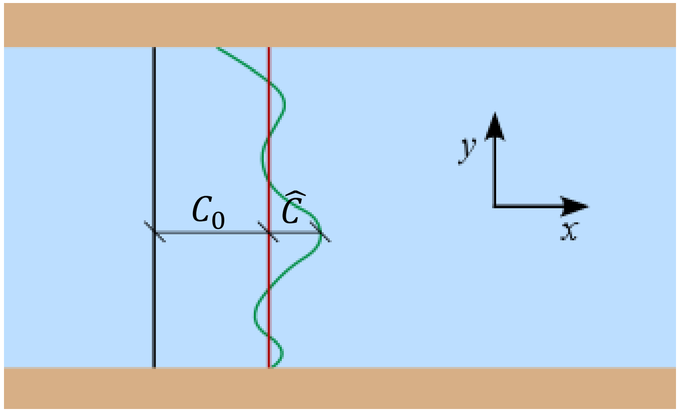
The simplification of the integrals is possible averaging the variables on the section, i.e.:

$$C_0 = \frac{\int_{-B/2}^{B/2} z_0 C \, \mathrm{d}y}{\int_{-B/2}^{B/2} z_0 \mathrm{d}y} = \frac{1}{A}\int_{-B/2}^{B/2} z_0 C \, \mathrm{d}y \qquad U_0 = \frac{\int_{-B/2}^{B/2} z_0 U_x C \mathrm{d}y}{\int_{-B/2}^{B/2} z_0 \mathrm{d}y} = \frac{Q}{A}$$

And applying the decomposition:



$U_x = U_0 + \widehat{U}$



$C = C_0 + \widehat{C}$

1D DISPERSION EQUATION

Noting that:

$$\int_{-B/2}^{B/2} z_0 U_x C \, \mathrm{d}y = \int_{-B/2}^{B/2} z_0 \left(U_0 + \hat{U}\,\right)\left(C_0 + \hat{C}\,\right) \mathrm{d}y$$

$$= \int_{-B/2}^{B/2} z_0 U_0 C_0 \, \mathrm{d}y + \int_{-B/2}^{B/2} \cancelto{0}{z_0 U_0 \hat{C}} \, \mathrm{d}y + \int_{-B/2}^{B/2} \cancelto{0}{z_0 \hat{U}\, C_0} \, \mathrm{d}y + \int_{-B/2}^{B/2} z_0 \hat{U}\, \hat{C} \, \mathrm{d}y$$

$$= A U_0 C_0 + \int_{-B/2}^{B/2} z_0 \hat{U}\, \hat{C} \, \mathrm{d}y = Q C_0 + \int_{-B/2}^{B/2} z_0 \hat{U}\, \hat{C} \, \mathrm{d}y$$

By developing the original equation:

$$\frac{\partial}{\partial t}(A C_0) + \frac{\partial}{\partial x}(Q C_0) = \frac{\partial}{\partial x} \int_{-B/2}^{B/2} z_0 k_x \frac{\partial C}{\partial x} \mathrm{d}y - \frac{\partial}{\partial x} \int_{-B/2}^{B/2} z_0 \hat{U}\, \hat{C} \, \mathrm{d}y$$

$$\longrightarrow \quad \frac{\partial}{\partial t}(A C_0) + \frac{\partial}{\partial x}(Q C_0) = \frac{\partial}{\partial x}\left(K_x A \frac{\partial C_0}{\partial x}\right)$$

<u>1D Dispersion Equation</u>

$$\overline{U_0 \hat{C}} = \frac{\int_{-B/2}^{B/2} z_0 U_0 \hat{C} \, \mathrm{d}y}{\int_{-B/2}^{B/2} z_0 \mathrm{d}y} = \frac{1}{A} \int_{-B/2}^{B/2} z_0 U_0 \hat{C} \, \mathrm{d}y = 0$$

$$\overline{\hat{U}\, C_0} = \frac{\int_{-B/2}^{B/2} z_0 \hat{U}\, C_0 \, \mathrm{d}y}{\int_{-B/2}^{B/2} z_0 \mathrm{d}y} = \frac{1}{A} \int_{-B/2}^{B/2} z_0 \hat{U}\, C_0 \, \mathrm{d}y = 0$$

where:

$$K_x \frac{\partial C_0}{\partial x} = -\frac{1}{A}\int_{-B/2}^{B/2} z_0 \left(\widehat{U}\,\widehat{C} - k_x \frac{\partial C}{\partial x}\right) \mathrm{dy}$$

<u>Longitudinal mixing coefficient</u>

Further simplification can be done by imposing the continuity equation, which is determined by integrating along y the 2D continuity equation:

$$\frac{\partial z_0}{\partial t} + \frac{\partial}{\partial x}(z_0 U_{x0}) + \frac{\partial}{\partial y}(z_0 U_{y0}) = 0 \xrightarrow{\int} \int_{-B/2}^{B/2} \frac{\partial z_0}{\partial t} \mathrm{d}y + \int_{-B/2}^{B/2} \frac{\partial (z_0 U_{x0})}{\partial x} \mathrm{d}y + [z_0 U_{y0}]_{\pm\frac{B}{2}} = 0$$

by Leibniz:

$$\frac{\partial}{\partial t}\int_{-B/2}^{B/2} z_0 \mathrm{d}y - \frac{1}{2} z_0 \frac{\partial B}{\partial t}\bigg|_{y=\frac{B}{2}} + \frac{1}{2} z_0 \frac{\partial B}{\partial t}\bigg|_{y=-\frac{B}{2}} + \frac{\partial}{\partial x}\int_{-B/2}^{B/2} z_0 U_{x0} \mathrm{d}y - \frac{1}{2} z_0 U_{x0} \frac{\partial B}{\partial x}\bigg|_{y=\frac{B}{2}}$$
$$+ \frac{1}{2} z_0 U_{x0} \frac{\partial B}{\partial x}\bigg|_{y=-\frac{B}{2}} + z_0 U_{y0}\Big|_{y=\frac{B}{2}} - z_0 U_{y0}\Big|_{y=-\frac{B}{2}} = 0$$

1D DISPERSION EQUATION

By grouping:

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}+z_0\left[\frac{1}{2}\frac{\partial B}{\partial t}+\cancelto{0}{\frac{1}{2}\frac{\partial B}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}}-\frac{\mathrm{d}y}{\mathrm{d}t}\right]_{y=-B/2}-z_0\left[\frac{1}{2}\frac{\partial B}{\partial t}+\cancelto{0}{\frac{1}{2}\frac{\partial B}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}}-\frac{\mathrm{d}y}{\mathrm{d}t}\right]_{y=B/2}=0$$

$$\longrightarrow \quad \frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0$$

<u>1D Continuity equation</u>

Expading the dispersion equation, we find:

$$C_0\frac{\partial A}{\partial t}+A\frac{\partial C_0}{\partial t}+C_0\frac{\partial Q}{\partial x}+Q\frac{\partial C_0}{\partial x}=\frac{\partial}{\partial x}\left(K_xA\frac{\partial C_0}{\partial x}\right)$$

$$A\frac{\partial C_0}{\partial t}+Q\frac{\partial C_0}{\partial x}+C_0\cancelto{0}{\left[\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}\right]}=\frac{\partial}{\partial x}\left(K_xA\frac{\partial C_0}{\partial x}\right)$$

3D continuity equation: $\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0$

2D continuity equation: $\frac{\partial z_0}{\partial t}+\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}=0$

1D continuity equation: $\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0$

By dividing for the section area A and considering implicitly section averaged variables:

$$\longrightarrow \quad \frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \frac{1}{A}\frac{\partial}{\partial x}\left(K_x A \frac{\partial C}{\partial x}\right)$$

This outcome is very useful. Why?

Let's assume uniform flow:
$$\begin{cases} \dfrac{\partial A}{\partial x} = 0 \\ \dfrac{\partial K_x}{\partial x} = 0 \end{cases}$$

Constant U means $\partial Q/\partial x = U\partial A/\partial x = 0$

The hydrodynamic conditions are the same along x.

Then: $$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = K_x\frac{\partial^2 C}{\partial x^2}$$

We can study the longitudinal dispersion according to a Fickian Model!

If we have a lumped mass insertion M in the section $x = 0$ at $t = 0$, the concentration is:

$$\longrightarrow \quad C(x,t) = \frac{M}{A\sqrt{4\pi K_x t}}\, e^{-\frac{(x-Ut)^2}{4K_x t}}$$

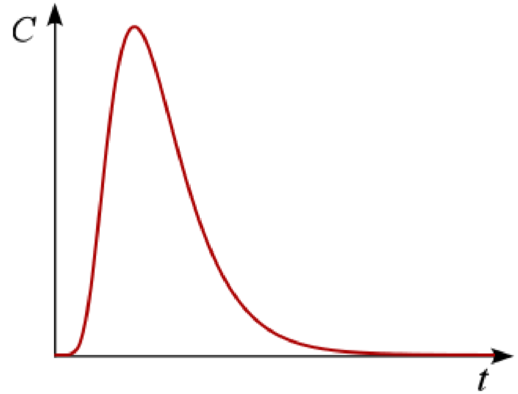
Taylor solution

TAYLOR SOLUTION

This solution implies that:

i. The concentration distribution $C(x_0, t)$ in a given section $x = x_0$ has skewness ($s \neq 0$):

$$\left|\frac{\partial C}{\partial t} > 0\right| > \left|\frac{\partial C}{\partial t} < 0\right|$$

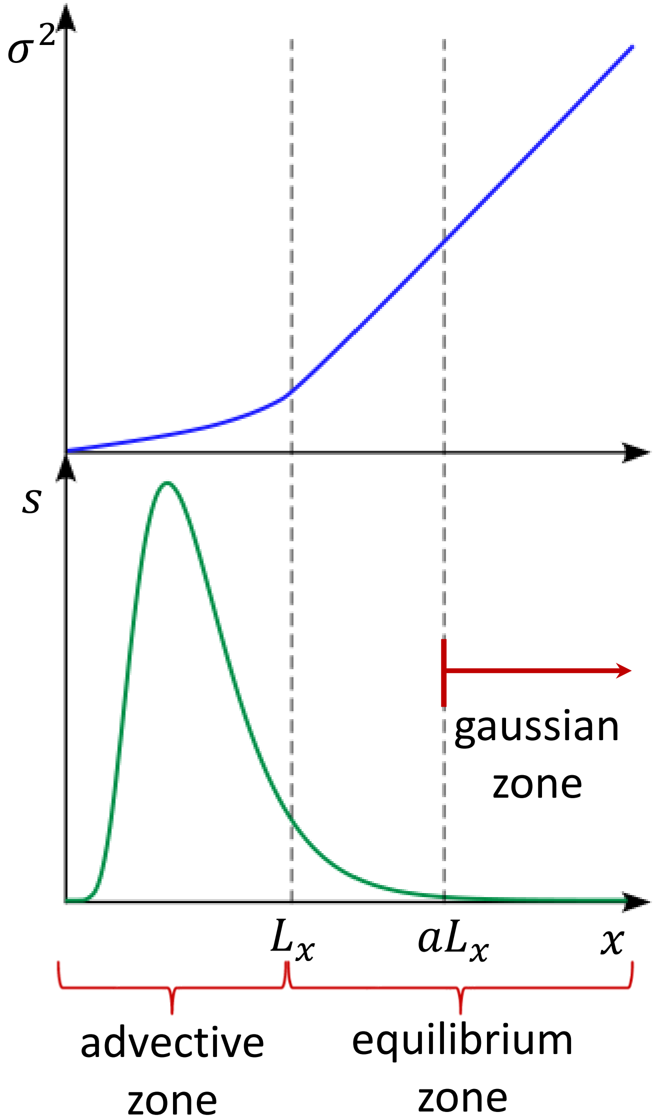


ii. For $t = t_0$, concentration $C(x, t_0)$ is Gaussian:
 - variance linearly increases with time, i.e. $\partial\sigma/\partial t = 2k_x$
 - odd statistical moments are zero.

Note that the Taylor solution is an ideal solution. In the real case the process reaches <u>asymptotically</u> this solution.

REGIMES OF DISPERSION

Experimental analyses recognize three zones.



i. $x < L_x$ **Advective zone**.
Variance increses less than linearly. Solute spreads out along firstly z and then y.
Skewness is large. Initially it monotonically increases until the peak due to non-uniform advection.

$$L_x = \kappa U \frac{B^2}{k_y} \qquad \kappa = 0.5 \div 0.6$$

ii. $x > L_x$ **Equilibrium zone**.
Dispersion processes reach the equilibrium. Non-uniform tranverse advection is balance with transverse mixing, i.e. $\sigma^2 \propto t$ and $\partial s / \partial t < 0$.

iii. $x > aL_x$ **Gaussian zone**.
Concentration follows Fickian model, i.e. $\sigma^2 \propto t$ and $s \cong 0$.

$\frac{B^2}{k_y} \rightarrow$ *It is the Eulerian time scale, i.e. the required average time by particle to visit the whole river section*

How much is the value of a?

The range is wide, because a is estimated by experimental studies and it strongly depends on the river path, section geometry and hydraulic regime of the investigated rivers.
In particular literature shows:

$a = 2.5$	(Fischer et al., 1979)
$a = 4 \div 5$	(Denton, 1990)
$a = 10$	(Sayre, 1968)
$a = 50$	(Liu & Cheng, 1980)

Difference of 1 order of magnitude

The greater values of a are due to the presence of wake zones along the river path. In this areas the velocity goes almost to zero, hence part of the solute can stay for long time in these zones increasing the skweness of the mean concentration distribution.

ESTIMATION OF K_x

There are several methods to correctly estimate the dispersion coefficient K_x in the Gaussian zone:

i. Empirical Formulas

ii. Chatwin Method

iii. Moments Method

These Methods have been described in the exercise lesson

iv. Calibration Method

v. Velocity Field Method

vi. Graphic Method

EMPIRICAL FORMULAS

The literature provides several formulas based on the multiple regression analysis of experimental data.

- McQuivey & Keefer (1974):

$$K_x = 0.058 \frac{Q}{i_b B}$$

- Liu (1977):

$$K_x = \alpha \frac{U^2}{z_0 u_*} \qquad \alpha = 0.18 \left(\frac{u_*}{U}\right)^{3/2}$$

- Seo & Cheong (1998)

$$K_x = 5.195\, z_0 u_* \left(\frac{B}{z_0}\right)^{0.62} \left(\frac{u_*}{U}\right)^{1.428}$$

- Kashelipour & Falconer (2002):

$$K_x = 10.612\, z_0 u_* \left(\frac{u_*}{U}\right)$$

CHATWIN METHOD

Chatwin rearranges the fundamental solution proposed by Taylor for 1D Fickian type process as following:

$$C(x,t) = \frac{M}{A\sqrt{4\pi K_x t}}\, e^{-\frac{(x-Ut)^2}{4K_x t}}$$

$$e^{\frac{(x-Ut)^2}{4K_x t}} = \frac{M}{A\sqrt{4\pi K_x}} \frac{1}{C\sqrt{t}}$$

$$R = \frac{M}{A\sqrt{4\pi K_x}}$$

ln

$$\frac{(x-Ut)^2}{4K_x t} = \ln\frac{R}{C\sqrt{t}} \longrightarrow \frac{(x-Ut)^2}{4K_x} = t\ln\frac{R}{C\sqrt{t}}$$

And then:

$$\frac{x}{2\sqrt{K_x}} - \frac{U}{2\sqrt{K_x}}t = \sqrt{t\ln\frac{R}{C\sqrt{t}}} = C^*$$

Fictitious concentration

CHATWIN METHOD

Often the amount of mass M is unknown. In this case it is useful rewritting the term R:

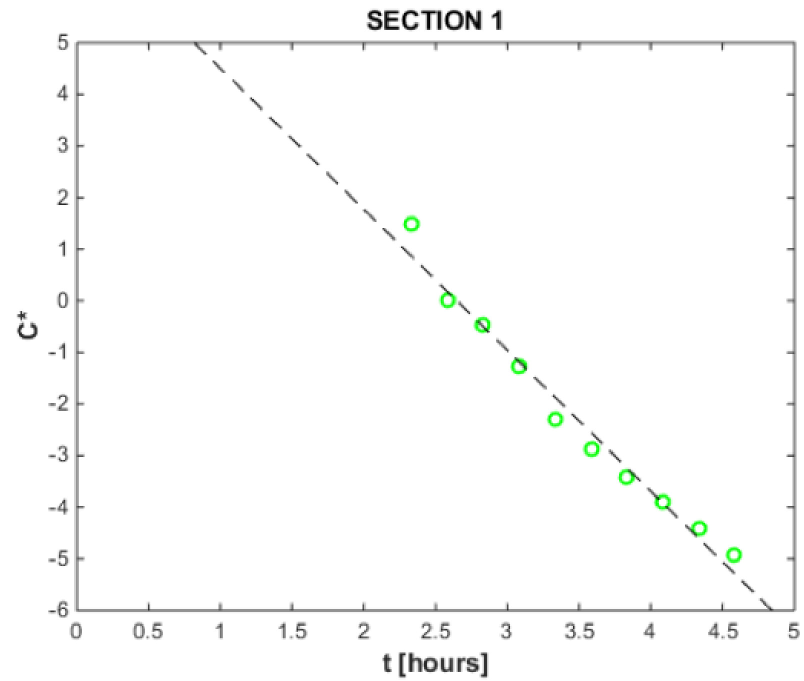
$$C_{max} = \frac{M}{A\sqrt{4\pi K_x t_{max}}}\, e^{\cancel{-\frac{(x-Ut_{max})^2}{4K_x t_{max}}}} \to 1 \quad \longrightarrow \quad C_{max} t_{max} = \frac{M}{A\sqrt{4\pi K_x}} = R$$

And finally:

$$C^* = -\frac{U}{2\sqrt{K_x}} t + \frac{x}{2\sqrt{K_x}}$$

with:

$$\begin{cases} C^* = \sqrt{t \cdot ln \dfrac{C_{max}\sqrt{t_{max}}}{C\sqrt{t}}} & t \le t_{max} \\ C^* = -\sqrt{t \cdot ln \dfrac{C_{max}\sqrt{t_{max}}}{C\sqrt{t}}} & t > t_{max} \end{cases}$$



K_x and U are calculated by the slope and the intercept of the dashed line

When the process of dispersion occurs in the Gaussian zone, the variance of the cloud increases linearly, i.e.:

$$\frac{\mathrm{d}{\sigma_x}^2}{\mathrm{d}t} = 2K_x$$

By applying the discretization method, it means:

$$K_x = \frac{1}{2}\frac{{\sigma_x}^2(t_2) - {\sigma_x}^2(t_1)}{t_2 - t_1}$$

Being:

$${\sigma_x}^2(t_i) = \frac{\int_{-\infty}^{+\infty}[x - \mu_x(t_i)]^2 C(x, t_i)\mathrm{d}t}{\int_{-\infty}^{+\infty} C(x, t_i)\mathrm{d}t}$$ → Statistical moment of order II

$$\mu_x(t_i) = \frac{\int_{-\infty}^{+\infty} xC(x, t_i)\mathrm{d}t}{\int_{-\infty}^{+\infty} C(x, t_i)\mathrm{d}t}$$ → Statistical moment of order I

In the practice, it is easier measuring $C(x_i, t)$ rather than $C(x, t_i)$. Hence we need to use the temporal variance of the concentration in x_i, ${\sigma_t}^2(x_i)$.

Fisher in 1966 demonstrated that:

$$K_x = \frac{1}{2} U_0 \frac{{\sigma_t}^2(x_2) - {\sigma_t}^2(x_1)}{\overline{t_2} - \overline{t_1}}$$

where

$$\overline{t_i} = \frac{A}{M} \int_0^{+\infty} t C(x_i, t) \mathrm{d}t$$

→ Statistical moment of order I. It is the time of the centroid in x_i

$${\sigma_t}^2(x_i) = \frac{A}{M} \int_0^{+\infty} [t - \overline{t_i}]^2 C(x_i, t) \mathrm{d}t$$

→ Statistical moment of order II

$$U_0 = \frac{x_2 - x_1}{\overline{t_2} - \overline{t_1}}$$

→ Mean velocity of the cloud centroid between x_1 and x_2