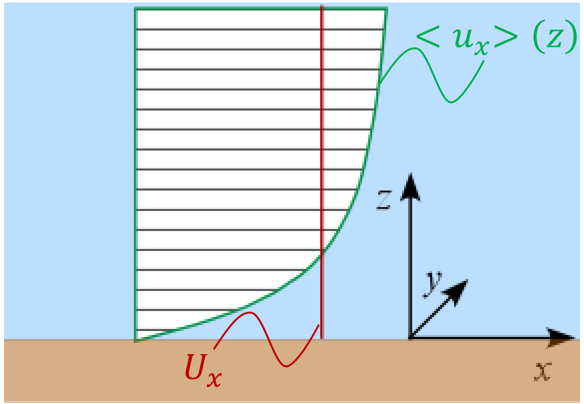
ICEA

UNIVERSITÀ DEGLI STUDI DI PADOVA

LESSON 10: 2D DISPERSION COEFFICIENTS



Assumptions:

- 1D flow, i.e. $< \boldsymbol{u} > = (< u_x > (z), 0, 0)$
  $< \boldsymbol{u} > \cdot \nabla < c > = < u_x > \partial < c > / \partial x$
- Wide rectangular section, i.e. $B \gg z_0$

The turbulent diffusion equation of the problem and the boundary conditions are:

$$\begin{cases} \dfrac{\partial < c >}{\partial t} + < u_x > \dfrac{\partial < c >}{\partial x} = \dfrac{\partial}{\partial x}\left(e_x \dfrac{\partial < c >}{\partial x} \right) + \dfrac{\partial}{\partial z}\left(e_z \dfrac{\partial < c >}{\partial z} \right) \\ e_z \dfrac{\partial < c >}{\partial z}\Bigg|_{z=0,z_0} = 0 \end{cases}$$

Flux is zero through the bottom and free surface

By the Reynolds analogy:

$$e_z(z) = u_* z_0 k \frac{z}{z_0}\left(1 - \frac{z}{z_0}\right) \longrightarrow \overline{e_z} = \frac{1}{z_0}\int_0^{z_0} e_z(z)\,\mathrm{d}z = 0.067 u_* z_0$$

Elder demonstrated that:

$$k_x = 5.86\, u_* z_0$$ → Elder coefficient

It is worth noting that:

$$\varepsilon = \frac{\overline{e_x}}{k_x} = \frac{0.067 u_* z_0}{5.86\, u_* z_0} \ll 1$$

$$\overline{e_x} = \overline{e_z}$$

Longitudinal dispersion is order of magnitude greater than longitudinal diffusion (and vertical diffusion, as well)

LONGITUDINAL MIXING COEFFICIENT

Let's analyze the time scale of each terms considering the "point of view" of the mean cloud centroid. In this case the equation of diffusion becomes:

$$\underbrace{\frac{\partial <c>}{\partial t}}_{1} + \underbrace{(<u_x> - U_x)\frac{\partial <c>}{\partial \xi}}_{2} = \underbrace{\frac{\partial}{\partial \xi}\left(e_x \frac{\partial <c>}{\partial \xi}\right)}_{3} + \underbrace{\frac{\partial}{\partial z}\left(e_z \frac{\partial <c>}{\partial z}\right)}_{4}$$

Where U_x is the vertical averaged velocity and (ξ, t) is the reference system centered in the mean cloud centroid.

Noting that:

(1) (3) → Classic diffusive processes

(2) (4) → Dispersive process due to non-uniform advection and vertical mixing

$$\frac{\partial c}{\partial t} \longrightarrow \frac{\partial c}{\partial t} - U_x \frac{\partial c}{\partial \xi}$$

$$\frac{\partial c}{\partial x} \longrightarrow \frac{\partial c}{\partial \xi}$$

The times scale of the problem are:

i. Vertical Mixing: $T_0 = \frac{{z_0}^2}{e_z}$

ii. Non-Uniform Advection: $T_1 = \frac{L_0}{U_x}$

iii. Longitudinal Dispersion: $T_2 = \frac{{L_0}^2}{k_x}$

By comparing the terms:

$$\frac{\partial < c > / \partial t}{\frac{\partial}{\partial z}\left(e_z \frac{\partial < c >}{\partial z}\right)} \propto \frac{k_x}{{L_0}^2}\frac{{z_0}^2}{e_z} = \frac{T_0}{T_2} \ll 1$$

$$\frac{(< u_x > - U_x)\, \partial < c > / \partial \xi}{\frac{\partial}{\partial z}\left(e_z \frac{\partial < c >}{\partial z}\right)} \propto \frac{U_x}{L_0}\frac{{z_0}^2}{e_z} = \frac{T_0}{T_1} \ll 1$$

$$\frac{\frac{\partial}{\partial \xi}\left(e_x \frac{\partial < c >}{\partial \xi}\right)}{\frac{\partial}{\partial z}\left(e_z \frac{\partial < c >}{\partial z}\right)} \propto \frac{e_x}{{L_0}^2}\frac{{z_0}^2}{e_z} = \left(\frac{z_0}{L_0}\right)^2 \ll 1$$

Vertical diffusivity is the fastest mechanisms

Indeed, in the natural systems: $T_0 \ll T_1 \ll T_2$

If we assume: $\begin{cases} \dfrac{T_0}{T_1} = \varepsilon \\ \dfrac{T_0}{T_2} = \varepsilon^2 \end{cases}$ $\xrightarrow{\text{yield}}$ $\dfrac{z_0}{L_0} = \varepsilon^{3/2}$

$$Pe = \frac{U_x z_0}{e_x} = \varepsilon^{-1/2} \gg 1$$

→ Peclet number (ratio between advection and diffusion). Diffusion is negligible!

The transport equation is:

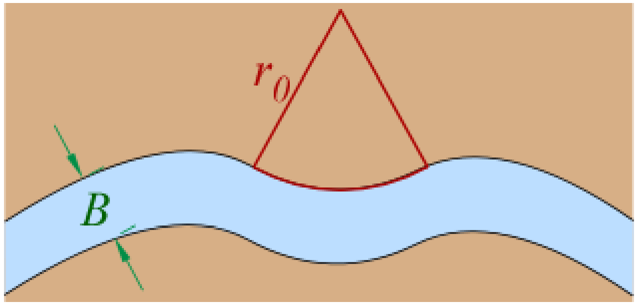
$$\underbrace{\frac{\partial < c >}{\partial t}}_{O(\varepsilon^2)} + \underbrace{(< u_x > - U_x)\frac{\partial < c >}{\partial \xi}}_{O(\varepsilon)} = \underbrace{\frac{\partial}{\partial \xi}\left(e_x \frac{\partial < c >}{\partial \xi}\right)}_{O(\varepsilon^3)} + \underbrace{\frac{\partial}{\partial z}\left(e_z \frac{\partial < c >}{\partial z}\right)}_{O(1)}$$

$$\varepsilon^2 = \frac{T_0}{T_2} = \frac{k_x}{e_z}\frac{{z_0}^2}{{L_0}^2} = \frac{1}{\varepsilon}\frac{{z_0}^2}{{L_0}^2} \longrightarrow \frac{z_0}{L_0} = \varepsilon^{3/2} \quad \textit{Since } \varepsilon = e_x/k_x \textit{ and } e_z = e_x$$

$$\varepsilon = \frac{T_0}{T_1} = \frac{U_x}{L_0}\frac{{z_0}^2}{e_z} = \frac{U_x z_0}{e_z}\frac{z_0}{L_0} = Pe\,\varepsilon^{3/2} \longrightarrow Pe = \varepsilon^{-1/2}$$

Similarly to transverse diffusivity, the transverse mixing coefficient k_y is estimated by means of experimental analysis.
k_y is strictly related to the curvature of the river path. In particular:

i. Straight channel $\left(\frac{BU_0}{u_* r_0} = 0\right)$: $0.15 < \frac{k_y}{u_* z_0} < 0.30$



ii. Weak curvature $\left(\frac{BU_0}{u_* r_0} < 2\right)$: $0.30 < \frac{k_y}{u_* z_0} < 0.90$

iii. meandering river $\left(\frac{BU_0}{u_* r_0} > 2\right)$: $1.00 < \frac{k_y}{u_* z_0} < 3.00$ → Sometimes it is greater than 3!

It is worth noting that for wide prismatic channel, we have:

$$\frac{k_y(y)}{u_*(y) z_0(y)} \cong \frac{k_y}{z_0\sqrt{g i z_0}} = \text{const} \longrightarrow k_y \propto z_0(y)^{3/2}$$

$$u_* = \sqrt{R_H g i}$$

$$R_H \cong z_0$$