ICEA

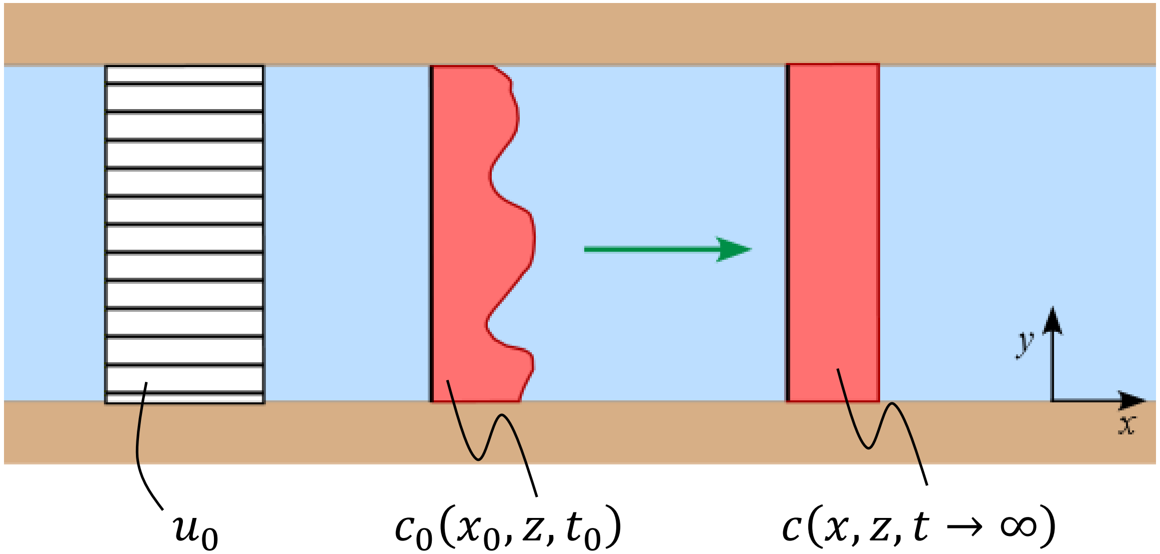
Università degli Studi di Padova

LESSON 09: 2D HYDRODYNAMIC DISPERSION

Hydrodynamic dispersion is given by:

i. Non-uniform convective flow
ii. Vertical and transverse diffusion

Considering **uniform flow**:

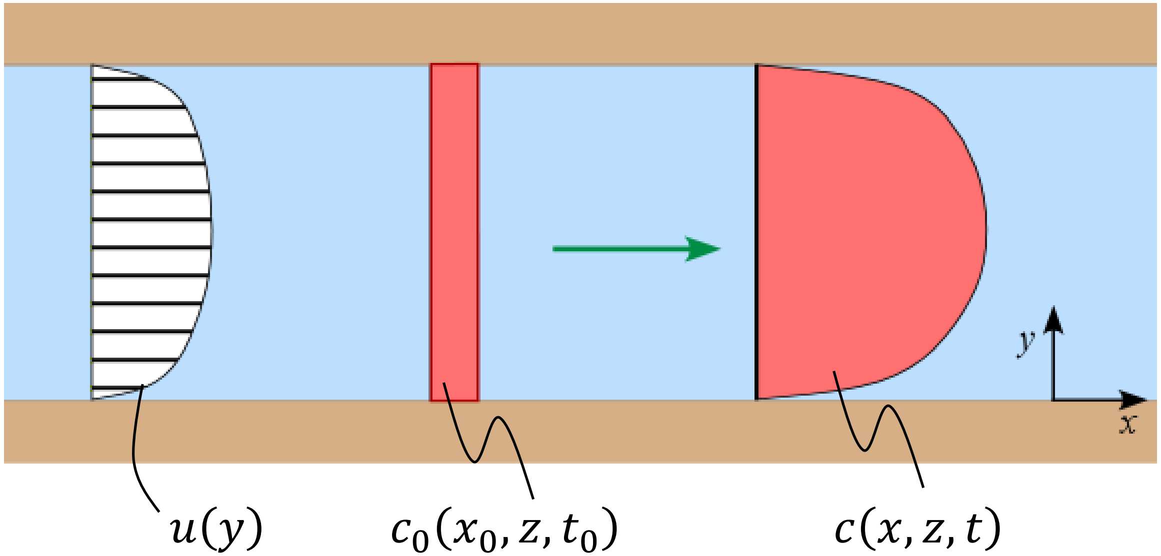


uniform flow
↓
No gradient
↓
No turbulent diffusion

Randomly the particles moves along the whole transversal section due to the molecular diffusivity D.

→ c becomes uniform along y.

Let's assume that the transport of solute is due to only **convective flow**:



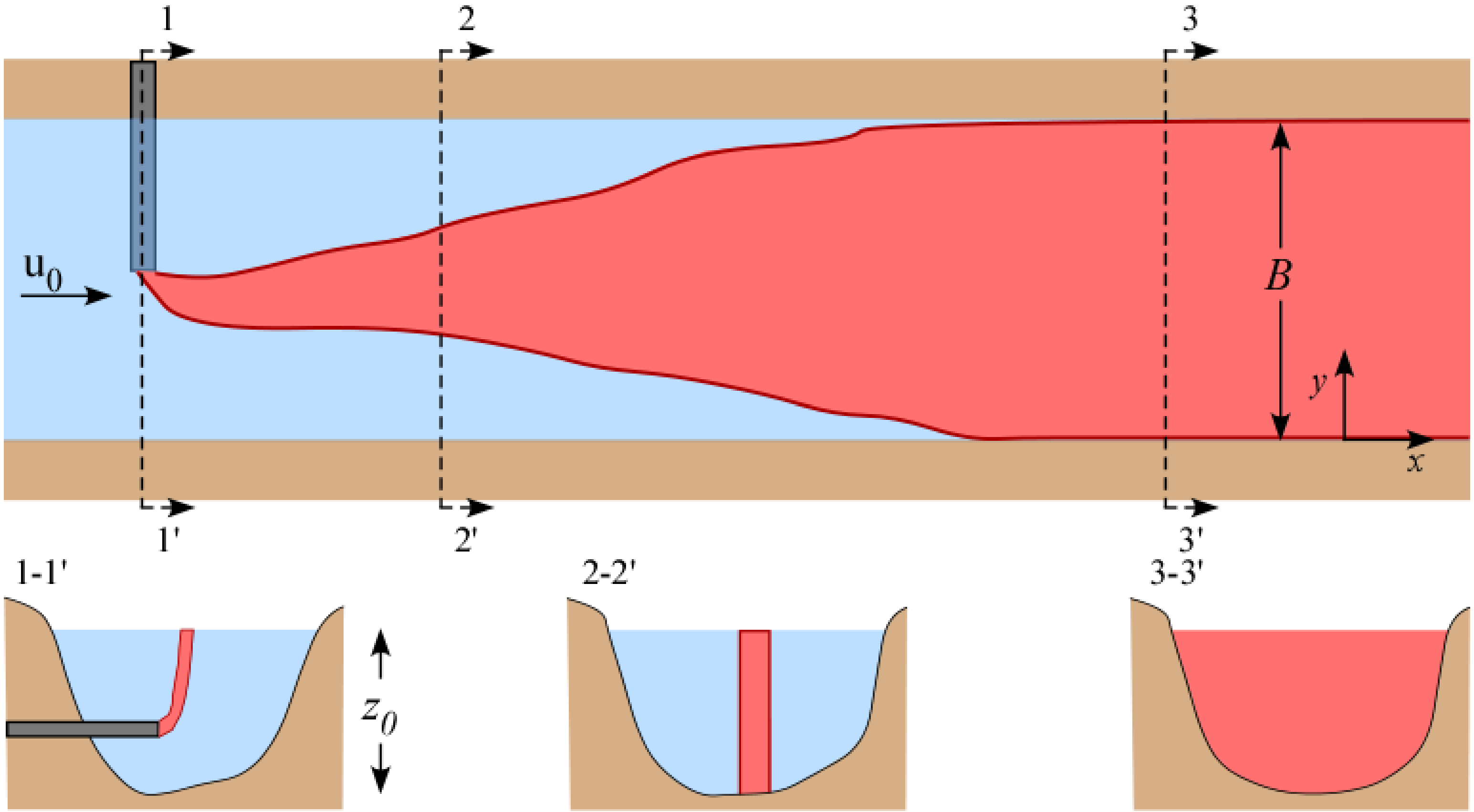
Velocity profile is parabolic, i.e. laminar flow

No turbulent diffusion

→ c distribution follows the velocity profile

If molecular diffusivity is neclected, the solute moves only along longitudinal direction.

Let's see the phenomenon in presence of **turbulent diffusion**.



$x < 20 \div 50\ z_0$

The mixing depends on <u>buoyancy</u> and <u>mixture momentum</u>

3D modelling

$20 \div 50 z_0 < x < 100 \div 300 B$

<u>Full vertical diffusion</u> of the solute. We can average on the vertical

2D modelling

$x > 100 \div 300\ B$

<u>Full transverse diffusion</u> of the solute. We can study 1D flow (c is uniform in the section)

1D modelling

INTRODUCTION

Being **Dispersion = Non uniform advection + Turbulent diffusion**, the process can be studied by moving the reference system following the centroid of the mean cloud.

↓

We can observe a random walk process of the solute which is ruled by a fictitious diffusive coefficient greater than the coefficients determined by the turbulence analysis!

N.B. This is true when the process of turbulent diffusion is fully developed!

We distinguish two kind of dispersion:

i. By averaging the mass balance along the vertical (2D flow shown in section 2-2'). The dispersion is expressed by the coefficients of longitudinal and transverse direction, k_x and k_y respectively.

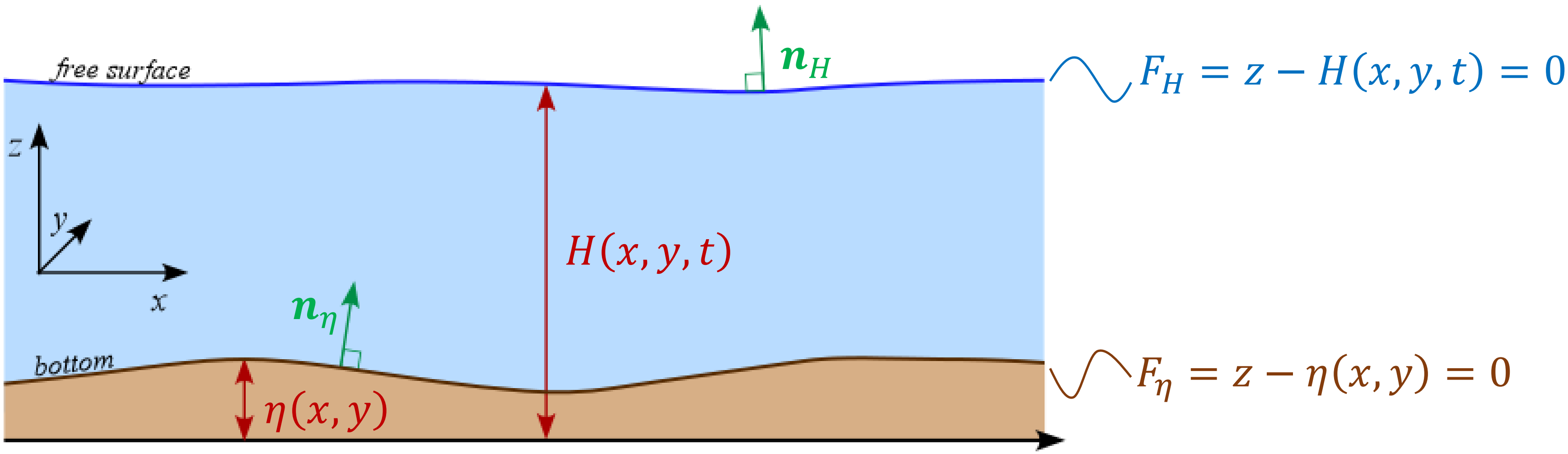
$$k_x = k_x(\partial U/\partial z\,, e_z) \qquad\qquad k_y = k_y(\partial U/\partial z\,, e_z)$$

i. By averaging the mass balance in the whole section (1D flow shown in section 3-3'). The process is expressed by the overall longitudinal dispersion coefficient K_x.

$$K_x = K_x\left(\partial U/\partial y\,, k_y\right)$$

Let's consider the figure below and the turbulent diffusion equation neglecting molecular diffusivity, i.e. $D\,\nabla^2 <c> = 0$:

$$\frac{\partial <c>}{\partial t} + \nabla \cdot (<\boldsymbol{u}><c>) = \frac{\partial}{\partial x}\left(e_x \frac{\partial <c>}{\partial x}\right) + \frac{\partial}{\partial y}\left(e_y \frac{\partial <c>}{\partial y}\right) + \frac{\partial}{\partial z}\left(e_z \frac{\partial <c>}{\partial z}\right)$$



If we consider the erosion and the deposition of sediment on the bottom, also the function η depends on time t:

$$\eta = \eta(x,y,t)$$

2D MASS BALANCE EQUATION

The <u>kinematic conditions</u> of the bottom and the free surface are:

i. $$\frac{\mathrm{d}F_\eta}{\mathrm{d}t} = 0 \quad \rightarrow \quad \left[\cancelto{0}{\frac{\partial \eta}{\partial t}} + \frac{\partial \eta}{\partial x}\cancelto{<u_x>}{\frac{\mathrm{d}x}{\mathrm{d}t}} + \frac{\partial \eta}{\partial y}\cancelto{<u_y>}{\frac{\mathrm{d}y}{\mathrm{d}t}} - \cancelto{<u_z>}{\frac{\mathrm{d}z}{\mathrm{d}t}}\right]_{z=\eta} = 0$$

ii. $$\frac{\mathrm{d}F_H}{\mathrm{d}t} = 0 \quad \rightarrow \quad \left[\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} < u_x > + \frac{\partial H}{\partial y} < u_y > - < u_z >\right]_{z=H} = 0$$

In the domain the mass is confined and it is constant, i.e.:

i. $[\boldsymbol{q} \cdot \boldsymbol{n}]_{z=\eta} = 0$

ii. $[\boldsymbol{q} \cdot \boldsymbol{n}]_{z=H} = 0$

The outward normal vector is defined as: $\boldsymbol{n} = \dfrac{\nabla F}{|\nabla F|}$

So in this case we have:

$$\mathbf{n}_\eta = \frac{(-\partial\eta/\partial x\,, -\partial\eta/\partial y\,, 1)}{\sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial y)^2}} \qquad \mathbf{n}_H = \frac{(-\partial H/\partial x\,, -\partial H/\partial y\,, 1)}{\sqrt{1 + (\partial H/\partial x)^2 + (\partial H/\partial y)^2}}$$

2D MASS BALANCE EQUATION

Moreover, the turbulent flux is defined as: $$\boldsymbol{q} = \left(e_x \frac{\partial < c >}{\partial x}, e_y \frac{\partial < c >}{\partial y}, e_z \frac{\partial < c >}{\partial z} \right)$$

The products between $\boldsymbol{n}$ and $\boldsymbol{q}$ give the dynamic conditions of the problem, that is:

i. $$\left[-e_x \frac{\partial < c >}{\partial x} \frac{\partial \eta}{\partial x} - e_y \frac{\partial < c >}{\partial y} \frac{\partial \eta}{\partial y} + e_z \frac{\partial < c >}{\partial z} \right]_{z=\eta} = 0$$

ii. $$\left[-e_x \frac{\partial < c >}{\partial x} \frac{\partial H}{\partial x} - e_y \frac{\partial < c >}{\partial y} \frac{\partial H}{\partial y} + e_z \frac{\partial < c >}{\partial z} \right]_{z=H} = 0$$

Now the problem is well posed and we can integrate the transport equation along z:

$$\int_\eta^H \frac{\partial < c >}{\partial t} \mathrm{dz} + \int_\eta^H \frac{\partial < c >< u_x >}{\partial x} \mathrm{dz} + \int_\eta^H \frac{\partial < c >< u_y >}{\partial y} \mathrm{dz} + \underbrace{\int_\eta^H \frac{\partial < c >< u_z >}{\partial z} \mathrm{dz}}_{1}$$

$$= + \int_\eta^H \frac{\partial}{\partial x} \left(e_x \frac{\partial < c >}{\partial x} \right) \mathrm{dz} + \int_\eta^H \frac{\partial}{\partial y} \left(e_y \frac{\partial < c >}{\partial y} \right) \mathrm{dz} + \underbrace{\int_\eta^H \frac{\partial}{\partial z} \left(e_z \frac{\partial < c >}{\partial z} \right) \mathrm{dz}}_{2}$$

Noting that:

1 $$\int_{\eta}^{H} \frac{\partial < c >< u_z >}{\partial z} \mathrm{dz} = [< c >< u_z >]_{z=H} - [< c >< u_z >]_{z=\eta}$$

2 $$\int_{\eta}^{H} \frac{\partial}{\partial z}\left(e_z \frac{\partial < c >}{\partial z}\right) \mathrm{dz} = \left[e_z \frac{\partial < c >}{\partial z}\right]_{z=H} - \left[e_z \frac{\partial < c >}{\partial z}\right]_{z=\eta}$$

And recalling the Leibniz rule of integration, we have:

$$\int_{\eta}^{H} \frac{\partial f}{\partial x} \mathrm{dz} = \frac{\partial}{\partial x} \int_{\eta}^{H} f \, \mathrm{dz} - f\Big|_{z=H} \frac{\partial H}{\partial x} + f\Big|_{z=\eta} \frac{\partial \eta}{\partial x}$$

By replacing (1) and (2) into the integrated transport equation, and by applying the Leibniz rule, we find:

$$\frac{\partial}{\partial t}\int_{\eta}^{H} < c > \mathrm{dz} - < c > \frac{\partial H}{\partial t} + < c > \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\int_{\eta}^{H} < c >< u_x > \mathrm{dz} - < c >< u_x > \frac{\partial H}{\partial x}$$

$$+ < c >< u_x > \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y}\int_{\eta}^{H} < c >< u_y > \mathrm{dz} - < c >< u_y > \frac{\partial H}{\partial y} + < c >< u_y > \frac{\partial \eta}{\partial y}$$

$$+ [< c >< u_z >]_{z=H} - [< c >< u_z >]_{z=\eta} = \frac{\partial}{\partial x}\int_{\eta}^{H} e_x \frac{\partial < c >}{\partial x} \mathrm{dz} - \left[e_x \frac{\partial < c >}{\partial x}\right]_{z=H} \frac{\partial H}{\partial x}$$

$$+ \left[e_x \frac{\partial < c >}{\partial x}\right]_{z=\eta} \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y}\int_{\eta}^{H} e_y \frac{\partial < c >}{\partial y} \mathrm{dz} - \left[e_y \frac{\partial < c >}{\partial y}\right]_{z=H} \frac{\partial H}{\partial y} + \left[e_y \frac{\partial < c >}{\partial y}\right]_{z=\eta} \frac{\partial \eta}{\partial y}$$

$$+ \left[e_z \frac{\partial < c >}{\partial z}\right]_{z=H} - \left[e_z \frac{\partial < c >}{\partial z}\right]_{z=\eta}$$

By grouping:

$$\frac{\partial}{\partial t}\int_{\eta}^{H} < c > \mathrm{dz} - < c > \left[\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} < u_x > \cancelto{0}{+ \frac{\partial H}{\partial y} < u_y >} - < u_z >\right]_{z=H} +$$

$$\frac{\partial}{\partial x}\int_{\eta}^{H} < c >< u_x > \mathrm{dz} + \frac{\partial}{\partial y}\int_{\eta}^{H} < c >< u_y > \mathrm{dz}$$

$$+ < c > \left[\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} < u_x > \cancelto{0}{+ \frac{\partial \eta}{\partial y} < u_y >} - < u_z >\right]_{z=\eta} = \frac{\partial}{\partial x}\int_{\eta}^{H} e_x \frac{\partial < c >}{\partial x}\mathrm{dz}$$

$$+ \frac{\partial}{\partial y}\int_{\eta}^{H} e_y \frac{\partial < c >}{\partial y}\mathrm{dz} + \left[-e_x \frac{\partial < c >}{\partial x}\frac{\partial H}{\partial x} \cancelto{0}{- e_y \frac{\partial < c >}{\partial y}}\frac{\partial H}{\partial y} + e_z \frac{\partial < c >}{\partial z}\right]_{z=H}$$

$$- \left[-e_x \frac{\partial < c >}{\partial x}\frac{\partial \eta}{\partial x} \cancelto{0}{- e_y \frac{\partial < c >}{\partial y}}\frac{\partial \eta}{\partial y} + e_z \frac{\partial < c >}{\partial z}\right]_{z=\eta}$$

And then:

$$\frac{\partial}{\partial t}\int_{\eta}^{H} < c > \mathrm{dz} + \frac{\partial}{\partial x}\int_{\eta}^{H} < c >< u_x > \mathrm{dz} + \frac{\partial}{\partial y}\int_{\eta}^{H} < c >< u_y > \mathrm{dz}$$
$$= \frac{\partial}{\partial x}\int_{\eta}^{H} e_x \frac{\partial < c >}{\partial x}\mathrm{dz} + \frac{\partial}{\partial y}\int_{\eta}^{H} e_y \frac{\partial < c >}{\partial y}\mathrm{dz}$$

The difference between free surface and the bottom is the water depth: $z_0(x,y) = H - \eta$. Then, the mean concentration along the vertical z is defined as:

$$C_0 = \frac{1}{z_0}\int_{\eta}^{H} < c > \ \mathrm{dz}$$

The other quantities of the equation, which are averaged along z, are expressed by overbar:

$$\overline{< c >< u_x >} = \frac{1}{z_0}\int_{\eta}^{H} < c >< u_x > \ dz \qquad \overline{< c >< u_y >} = \frac{1}{z_0}\int_{\eta}^{H} < c >< u_y > \ dz$$

$$\overline{e_x \frac{\partial < c >}{\partial x}} = \frac{1}{z_0}\int_{\eta}^{H} e_x \frac{\partial < c >}{\partial x} \ dz \qquad \overline{e_y \frac{\partial < c >}{\partial y}} = \frac{1}{z_0}\int_{\eta}^{H} e_y \frac{\partial < c >}{\partial y} \ dz$$

2D MASS BALANCE EQUATION

These definitions of the averaged quantities into the transport equation yield:

$$\frac{\partial}{\partial t}(z_0 C_0) + \frac{\partial}{\partial x}(z_0 \overline{< c >< u_x >}) + \frac{\partial}{\partial y}(z_0 \overline{< c >< u_y >})$$

$$= \frac{\partial}{\partial x}\left(z_0 \overline{e_x \frac{\partial < c >}{\partial x}}\right) + \frac{\partial}{\partial y}\left(z_0 \overline{e_y \frac{\partial < c >}{\partial y}}\right)$$

Let's see the variables along z. We can decompose them following the Reynolds approach:

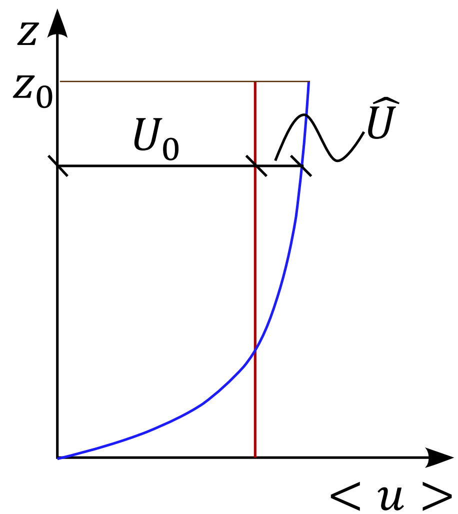
$$< c > = C_0 + \hat{C}$$

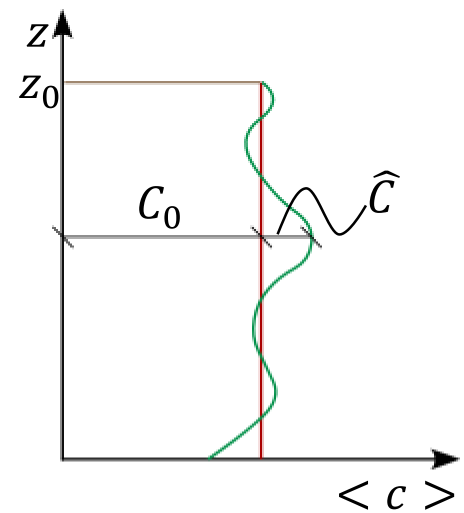
$$< u_x > = U_{x0} + \widehat{U}_x$$

$$< u_y > = U_{y0} + \widehat{U}_y$$

$$e_x = e_{x0} + \hat{e}_x$$

$$e_y = e_{y0} + \hat{e}_y$$





2D MASS BALANCE EQUATION

Replacing decomposed variables into vertical averaged products yield:

$$\overline{<c><u_x>} = C_0 U_{x0} + \cancelto{0}{\overline{\hat{C} U_{x0}}} + \cancelto{0}{\overline{C_0 \hat{U}_x}} + \overline{\hat{C}\hat{U}_x} = C_0 U_{x0} + \overline{\hat{C}\hat{U}_x}$$

$$\overline{<c><u_y>} = C_0 U_{y0} + \cancelto{0}{\overline{\hat{C} U_{y0}}} + \cancelto{0}{\overline{C_0 \hat{U}_y}} + \overline{\hat{C}\hat{U}_y} = C_0 U_{y0} + \overline{\hat{C}\hat{U}_y}$$

$$\overline{e_x \frac{\partial <c>}{\partial x}} = e_{x0}\frac{\partial C_0}{\partial x} + \cancelto{0}{\overline{\hat{e}_x \frac{\partial C_0}{\partial x}}} + \cancelto{0}{\overline{e_{x0}\frac{\partial \hat{C}}{\partial x}}} + \overline{\hat{e}_x \frac{\partial \hat{C}}{\partial x}} = e_{x0}\frac{\partial C_0}{\partial x} + \overline{\hat{e}_x \frac{\partial \hat{C}}{\partial x}}$$

$$\overline{e_y \frac{\partial <c>}{\partial y}} = e_{y0}\frac{\partial C_0}{\partial y} + \cancelto{0}{\overline{\hat{e}_y \frac{\partial C_0}{\partial y}}} + \cancelto{0}{\overline{e_{y0}\frac{\partial \hat{C}}{\partial y}}} + \overline{\hat{e}_y \frac{\partial \hat{C}}{\partial y}} = e_{y0}\frac{\partial C_0}{\partial y} + \overline{\hat{e}_y \frac{\partial \hat{C}}{\partial y}}$$

Then:

$$\frac{\partial}{\partial t}(z_0 C_0) + \frac{\partial}{\partial x}(z_0 C_0 U_{x0}) + \frac{\partial}{\partial x}\left(z_0 \overline{\hat{C}\hat{U}_x}\right) + \frac{\partial}{\partial y}(z_0 C_0 U_{y0}) + \frac{\partial}{\partial y}\left(z_0 \overline{\hat{C}\hat{U}_y}\right)$$

$$= \frac{\partial}{\partial x}\left(z_0 e_{x0}\frac{\partial C_0}{\partial x}\right) + \frac{\partial}{\partial x}\left(z_0 \overline{\hat{e}_x \frac{\partial \hat{C}}{\partial x}}\right) + \frac{\partial}{\partial y}\left(z_0 e_{y0}\frac{\partial C_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(z_0 \overline{\hat{e}_y \frac{\partial \hat{C}}{\partial y}}\right)$$

2D MASS BALANCE EQUATION

Finally:

$$\frac{\partial}{\partial t}(z_0 C_0) + \frac{\partial}{\partial x}(z_0 C_0 U_{x0}) + \frac{\partial}{\partial y}(z_0 C_0 U_{y0}) = \frac{\partial}{\partial x}\left[z_0\left(e_{x0}\frac{\partial C_0}{\partial x} + \overline{\hat{e}_x \frac{\partial \hat{C}}{\partial x}} - \overline{\hat{C}\widehat{U}_x}\right)\right] +$$

$$+ \frac{\partial}{\partial y}\left[z_0\left(\underbrace{e_{y0}\frac{\partial C_0}{\partial y}}_{\textit{Turbulent Diffusion}} + \underbrace{\overline{\hat{e}_y \frac{\partial \hat{C}}{\partial y}} - \overline{\hat{C}\widehat{U}_y}}_{\textit{Dispersion}}\right)\right]$$

When the solute is completely mixed along z (section 2-2' in figure 3), i.e. $t > {z_0}^2/e_x$, the non-uniform advection and the vertical gradient of turbulent diffusion are in equilibrium:

$$\overline{\hat{e}_x \frac{\partial \hat{C}}{\partial x}} - \overline{\hat{C}\widehat{U}_x} = k_x \frac{\partial C_0}{\partial x} \longrightarrow$$

k_x is the <u>longitudinal dispersion coefficient</u>

$$\overline{\hat{e}_y \frac{\partial \hat{C}}{\partial y}} - \overline{\hat{C}\widehat{U}_y} = k_y \frac{\partial C_0}{\partial x} \longrightarrow$$

k_y is the <u>transverse dispersion coefficient</u>

2D DISPERSION EQUATION

Dispersion coefficient is much greater than diffusion coefficient, thus the tranport equation can be simplified in the following:

$$\longrightarrow \quad \frac{\partial}{\partial t}(z_0 C_0) + \frac{\partial}{\partial x}(z_0 C_0 U_{x0}) + \frac{\partial}{\partial y}(z_0 C_0 U_{y0}) = \frac{\partial}{\partial x}\left[z_0 k_x \frac{\partial C_0}{\partial x}\right] + \frac{\partial}{\partial y}\left[z_0 k_y \frac{\partial C_0}{\partial y}\right]$$

This equation is useful to solve numerically dispersion problem. For determining analytical solution the latter needs of being simplified by using the continuity equation.

2D continuity equation for uncompressible fluid is:

$$\longrightarrow \quad \frac{\partial z_0}{\partial t} + \frac{\partial}{\partial x}(z_0 U_{x0}) + \frac{\partial}{\partial y}(z_0 U_{y0}) = 0$$

Then, expanding the dispersion equation, we find:

$$C_0 \frac{\partial z_0}{\partial t} + z_0 \frac{\partial C_0}{\partial t} + C_0 \frac{\partial}{\partial x}(z_0 U_{x0}) + (z_0 U_{x0}) \frac{\partial C_0}{\partial x} + C_0 \frac{\partial}{\partial y}(z_0 U_{y0}) + (z_0 U_{y0}) \frac{\partial C_0}{\partial y}$$
$$= \frac{\partial}{\partial x}\left[z_0 k_x \frac{\partial C_0}{\partial x}\right] + \frac{\partial}{\partial y}\left[z_0 k_y \frac{\partial C_0}{\partial y}\right]$$

2D DISPERSION EQUATION

Rearranging the equation: $$\frac{\partial C_0}{\partial t} + U_{x0}\frac{\partial C_0}{\partial x} + U_{y0}\frac{\partial C_0}{\partial y} + C_0 \cancelto{0}{\left[\frac{\partial z_0}{\partial t} + \frac{\partial}{\partial x}(z_0 U_{x0}) + \frac{\partial}{\partial y}(z_0 U_{y0})\right]}$$
$$= \frac{1}{z_0}\frac{\partial}{\partial x}\left[z_0 k_x \frac{\partial C_0}{\partial x}\right] + \frac{1}{z_0}\frac{\partial}{\partial y}\left[z_0 k_y \frac{\partial C_0}{\partial y}\right]$$

Finally, considering implicitly vertical averaged variables we have:

$$\longrightarrow \quad \frac{\partial C}{\partial t} + U_x\frac{\partial C}{\partial x} + U_y\frac{\partial C}{\partial y} = \frac{1}{z_0}\frac{\partial}{\partial x}\left[z_0 k_x \frac{\partial C}{\partial x}\right] + \frac{1}{z_0}\frac{\partial}{\partial y}\left[z_0 k_y \frac{\partial C}{\partial y}\right]$$

Integrating $\nabla \cdot \boldsymbol{u} = 0$ *in* z:

$$\int_\eta^H \frac{\partial U_{x0}}{\partial x}\,\mathrm{dz} + \int_\eta^H \frac{\partial U_{y0}}{\partial y}\,\mathrm{dz} + \int_\eta^H \frac{\partial U_{z0}}{\partial z}\,\mathrm{dz} = 0$$

$$\xrightarrow{\textit{Leibniz}} \quad \frac{\partial}{\partial x}\int_\eta^H U_{x0}\,\mathrm{dz} - U_{x0}\frac{\partial H}{\partial x} + U_{x0}\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y}\int_\eta^H U_{y0}\,\mathrm{dz} - U_{y0}\frac{\partial H}{\partial y} + U_{y0}\frac{\partial \eta}{\partial y} + U_{z0}\Big|_\eta^H = 0$$

$$\frac{\partial[U_{x0}(H-\eta)]}{\partial x} + \frac{\partial[U_{y0}(H-\eta)]}{\partial y} + \underbrace{\left[U_{z0} - U_{x0}\frac{\partial H}{\partial x} - U_{y0}\frac{\partial H}{\partial y}\right]_{z=H}}_{\partial H/\partial t} - \underbrace{\left[U_{z0} - U_{x0}\frac{\partial \eta}{\partial x} - U_{y0}\frac{\partial \eta}{\partial y}\right]_{z=\eta}}_{\partial \eta/\partial t} = 0$$

$$\frac{\partial(U_{x0}z_0)}{\partial x} + \frac{\partial(U_{y0}z_0)}{\partial y} + \frac{\partial(H-\eta)}{\partial t} = 0 \quad \longrightarrow \quad \frac{\partial z_0}{\partial t} + \frac{\partial}{\partial x}(U_{x0}z_0) + \frac{\partial}{\partial y}(U_{y0}z_0) = 0$$