ICEA

Università degli Studi di Padova

LESSON 8: TURBULENT DIFFUSION COEFFICIENTS

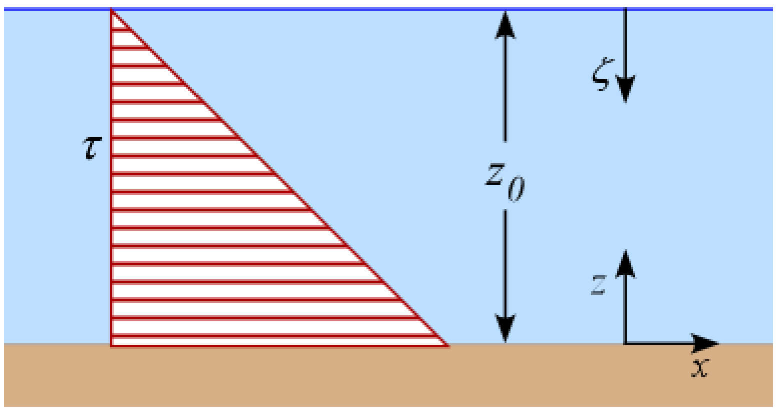
REYNOLDS ANALOGY FOR EVALUATING e_i

Reynolds analogy says that: $\nu_T \propto u_* z_0 \longleftrightarrow e_z \propto u_* z_0$

- ν_T: Kinematic viscosity (Momentum diffusivity)
- u_*: Friction velocity
- e_z: Turbulent diffusivity

$\longrightarrow$ $\nu_T = e_z$

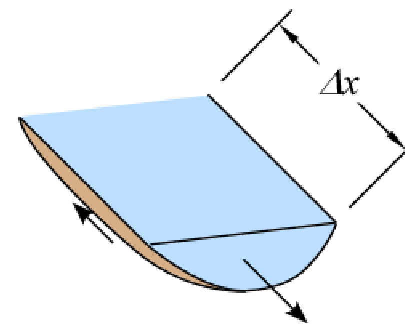
Let's study ν_T, assuming uniform flow, i.e. $\boldsymbol{u} = (u, 0, 0)$.



For uniform flow, the force balance gives:

$$\gamma\, A\, \Delta x\, i = P \tau_0\, \Delta x$$

- A: cross sectional area
- P: wetted perimeter
- i: bottom slope



$\longrightarrow$ $\tau_0 = \frac{A}{P}\gamma i = R_H \gamma i$ Bottom shear stress

R_H: Hydraulic Radius

REYNOLDS ANALOGY FOR EVALUATING e_i

The friction velocity is defined as:

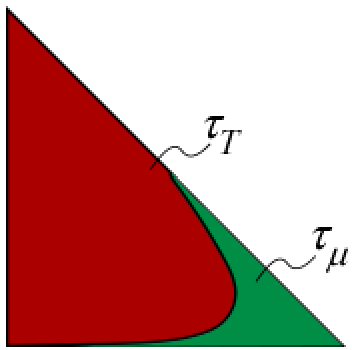
$$u_*^2 = \frac{\tau_0}{\rho} = \frac{\gamma}{\rho} R_H i = g R_H i \xrightarrow{B \gg z_0} u_*^2 = g z_0 i$$

Along the vertical the shear stress increases linearly, in according to the latter expression:

$$\tau = \gamma i \zeta \longrightarrow \frac{\tau}{\rho} = g i \zeta = u_*^2 \frac{\zeta}{z_0}$$

$$g i = \frac{u_*^2}{z_0}$$

The total shear stress along the vertical is due to the sum of the viscous shear stress τ_μ and the turbulent shear stress τ_T:



$$\tau = \tau_\mu + \tau_T$$

$$\tau = \mu \frac{\mathrm{d}U}{\mathrm{dz}} - \rho < u'_x u'_z >$$

$$U = < u >$$

$$\tau = \mu \frac{\mathrm{d}U}{\mathrm{dz}} + \rho \nu_T \frac{\mathrm{d}U}{\mathrm{dz}} \longrightarrow \tau \cong \rho \nu_T \frac{\mathrm{d}U}{\mathrm{dz}}$$

Negligible when $z \gg \delta^*$ (viscous sublayer)

For a rectangular channel of width B: $R_H = \frac{B z_0}{B + 2z_0} = \frac{z_0}{1 + 2z_0/B} \xrightarrow{B \gg z_0} R_H \cong z_0$

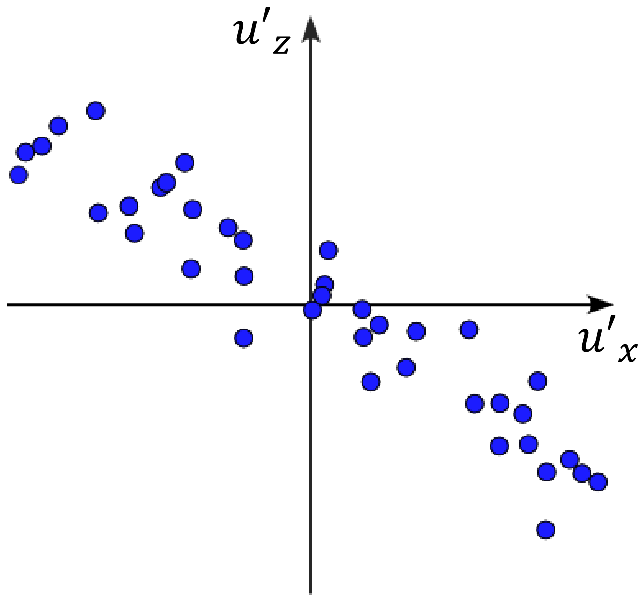
By expliciting ν_T, and by replacing the expression of τ/ρ that we found previously, we have:

$$\nu_T = \frac{\tau/\rho}{\mathrm{d}U/\mathrm{dz}} = \frac{{u_*}^2\zeta}{(\mathrm{d}U/\mathrm{dz})z_0} = \frac{{u_*}^2}{\mathrm{d}U/\mathrm{dz}}\frac{z_0 - z}{z_0} \qquad \zeta = z_0 - z$$

The turbulent viscosity depends on z and $\mathrm{d}U/\mathrm{d}z$. The velocity gradient is known when the flow is turbulent because the velocity profile can be assumed logarithmic.

To determine $\mathrm{d}U/\mathrm{d}z$ we adopt the Prandtl Solution, that assumes:

- $\tau = -\rho < u'_x u'_z > \cong \tau_0$
- $|u'_x| = l\,|\mathrm{d}U/\mathrm{dz}|$
- $|u'_x| \sim |u'_z|$
- $u'_x > 0 \quad \rightarrow \quad u'_z < 0$



Where:

$l = k\,z$ Mixing Length

$k \cong 0.41$ Von Karman Constant

By these assumptions:

$$\tau \cong \tau_0 = \rho l^2 \left(\frac{\mathrm{d}U}{\mathrm{dz}}\right)^2 = \rho (kz)^2 \left(\frac{\mathrm{d}U}{\mathrm{dz}}\right)^2$$

Hence the friction velocity can be expressed as:

$$u_*{}^2 = \frac{\tau_0}{\rho} = (kz)^2 \left(\frac{\mathrm{d}U}{\mathrm{dz}}\right)^2 \longrightarrow \frac{\mathrm{d}U}{\mathrm{dz}} = \frac{u_*}{kz}$$

This estimation of the velocity gradient into the definition of ν_T yields:

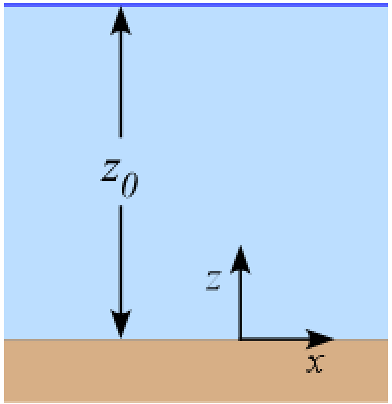
$$\nu_T = \frac{u_*{}^2}{\mathrm{d}U/\mathrm{dz}} \frac{z_0 - z}{z_0}$$

$$\nu_T = u_* kz \frac{z_0 - z}{z_0} \longrightarrow e_z = \nu_T = u_* z_0 k \frac{z}{z_0}\left(1 - \frac{z}{z_0}\right)$$

$\nu_T, e_z \propto z^2$

Parabolic distribution!

VERTICAL TURBULENT DIFFUSIVITY



By Reynolds analogy:

$$\nu_T = e_z \implies e_z(z) = k u_* z_0 \frac{z}{z_0}\left(1 - \frac{z}{z_0}\right)$$

It has been confirmed by several experimental studies

In many cases it is not important the distribution of e_z, but only its mean value $\bar{e_z}$, that is defined as:

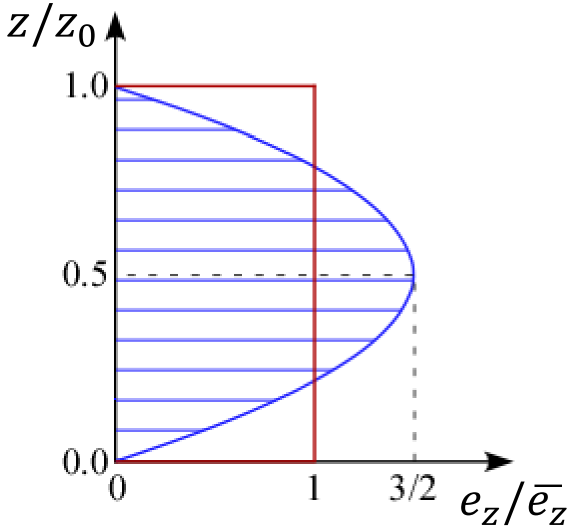
$$\bar{e_z} = \frac{1}{z_0}\int_0^{z_0} e_z(z)\,\mathrm{d}z$$

that is:

$$\bar{e_z} = \frac{k u_*}{z_0}\int_0^{z_0} z\,\mathrm{d}z - \frac{k u_*}{{z_0}^2}\int_0^{z_0} z^2\,\mathrm{d}z$$

$$\bar{e_z} = \frac{k u_*}{z_0}\left(\frac{{z_0}^2}{2}\right) - \frac{k u_*}{{z_0}^2}\left(\frac{{z_0}^3}{3}\right) = k u_*\left(\frac{z_0}{2} - \frac{z_0}{3}\right) = \frac{k u_* z_0}{6}$$

VERTICAL TURBULENT DIFFUSIVITY



$$\bar{e_z} = 0.067\, u_* z_0$$

Vertical turbulent diffusivity

There are two main issues when we analyze the vertical diffusivity in rivers:

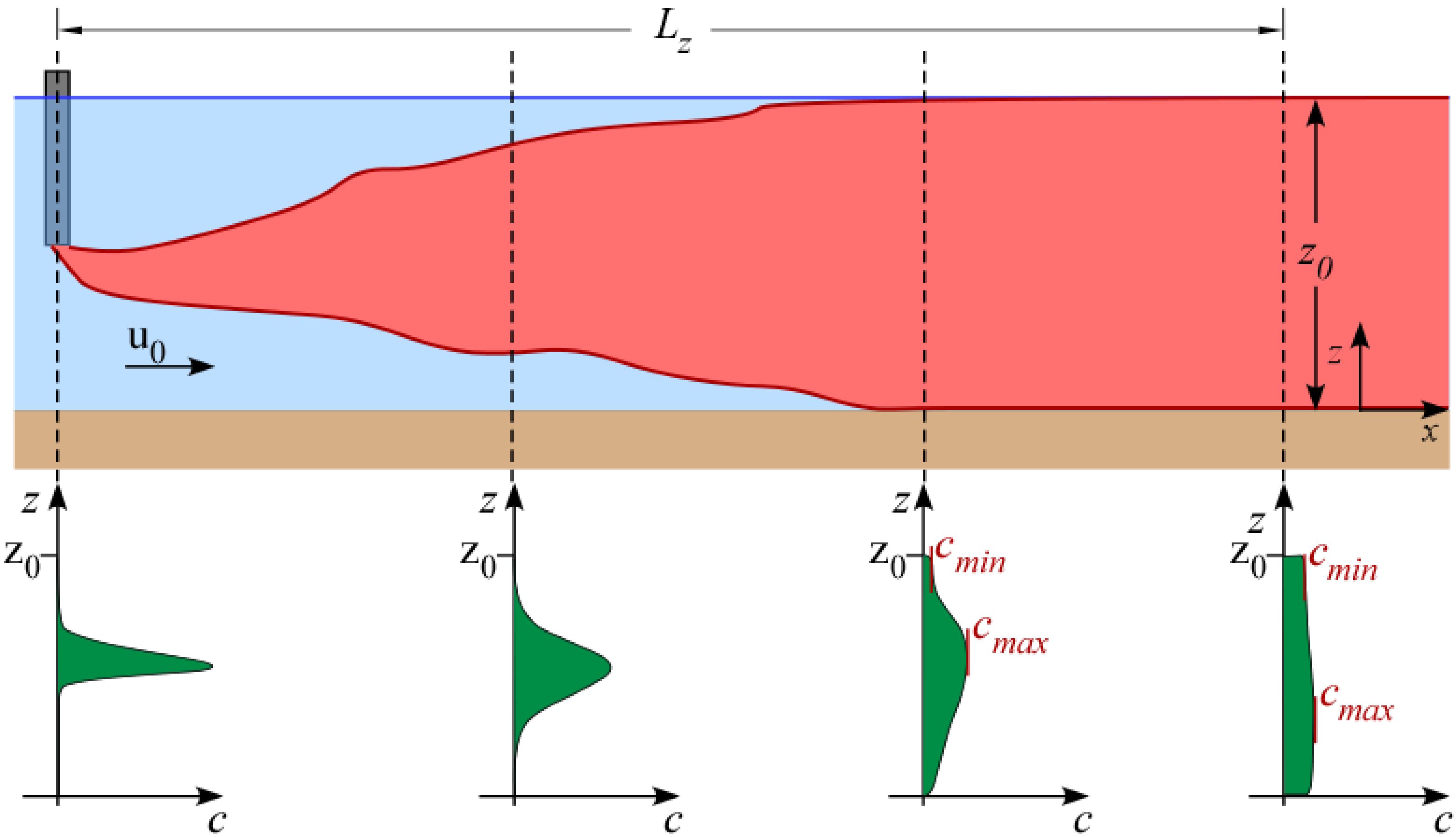
i. Large difference between the specific weight of water and solute, i.e. $|\gamma_s - \gamma| \gg 1$

ii. Irregular path of the river, that causes secondary currents and thus it influences the vertical diffusivity

In these case the approximation suggested by the Reynolds analogy is not valid anymore. It is useful to adopt empirical formulation to estimate e_z (hereinafter it is the mean value of diffusivity) by *in situ* measurements.

We measure the distance from the insertion L_z, such that, along z, $c_{min}/c_{max} = 0.98$

$$\rightarrow \quad e_z = \beta \frac{u_0 {z_0}^2}{L_z} \qquad \beta = 0.134$$



Assumptions:

- Negligible transversal velocity gradient
- Negligible transversal concentration gradient
- Uniform 1D flow, i.e. $\boldsymbol{u} = (u_0, 0, 0)$

We can average the concentration along the transversal direction y, that is:

$$\bar{c}(x,z) = \frac{1}{B}\int_{-B/2}^{B/2} c(x,y,z)\,\mathrm{d}y$$

Channel width (B)

The A-D equation can be now approximated by:

$$u_0 \frac{\partial \bar{c}}{\partial x} = \frac{\partial}{\partial z}\left(e_z \frac{\partial \bar{c}}{\partial z}\right)$$

It is true for the bulk flow. near the sides of the channel it is not valid anymore.

THE CONCENTRATION DISTRIBUTION METHOD

For lumped insertion of the solute mass m in the position z_{in} between $[0, z_0]$, and by assuming $e_z = \text{const}$, the solution of this equation is:

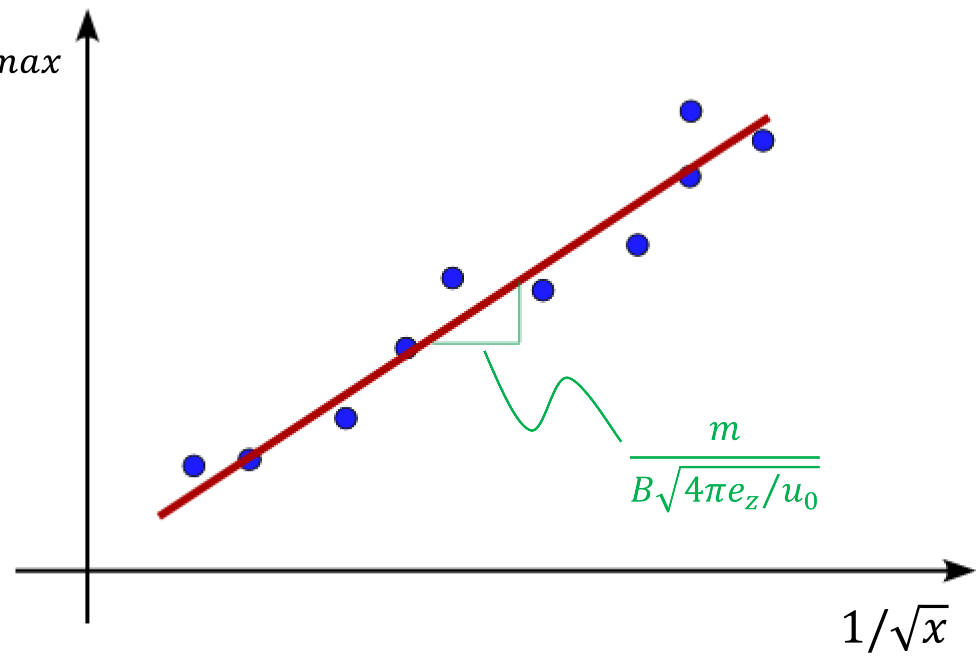
$$\bar{c}(x,z) = \frac{m}{B\sqrt{4\pi e_z\, x/u_0}}\, e^{-\frac{u_0(z-z_{in})^2}{4e_z x}}$$

This solution is valid until the solute does not reach the free surface ($z = z_0$) or the bottom ($z = 0$)

At $z \cong z_{in}$ we can record along x the maximum concentration of the section, that is:

$$\bar{c}_{max}(x) \cong \frac{m}{B\sqrt{4\pi e_z\, x/u_0}}$$

e_z is the only unknown parameter. We can determine it through the regression line of the experimental data collected in situ!



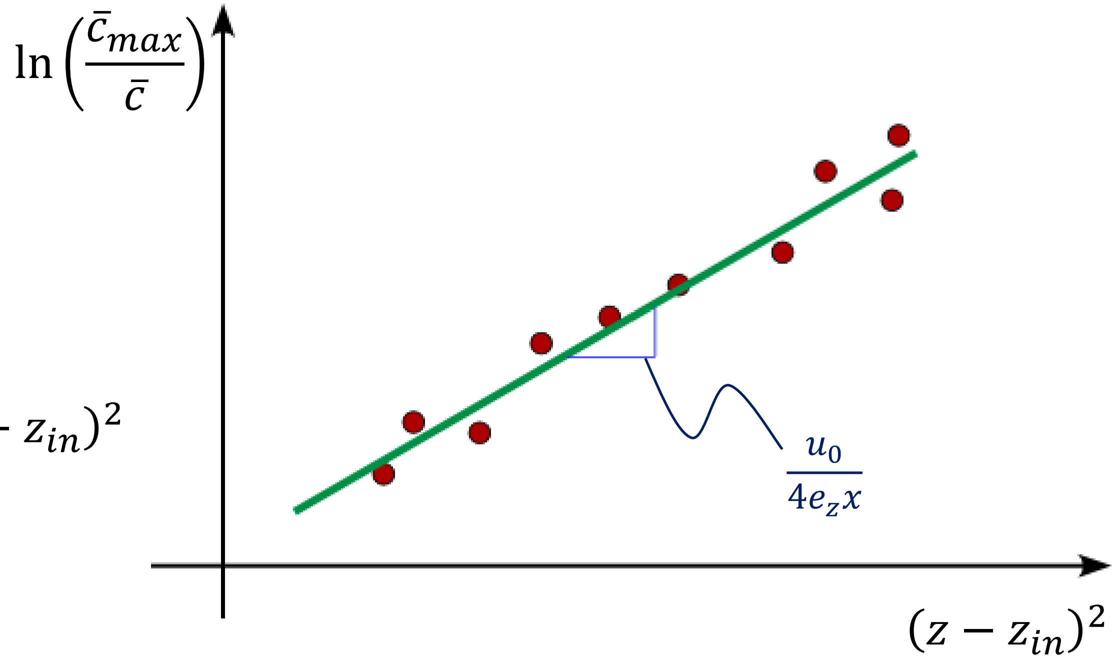
THE CONCENTRATION DISTRIBUTION METHOD

If we do not know the amount of m, we have to record the concentration along z in a given section located x far from the insertion point. The fundamental solution can be rearranged as following:

$$\bar{c} = \bar{c}_{max}\, e^{-\frac{u_0 (z-z_{in})^2}{4 e_z x}}$$

$$\frac{\bar{c}_{max}}{\bar{c}} = e^{\frac{u_0 (z-z_{in})^2}{4 e_z x}}$$

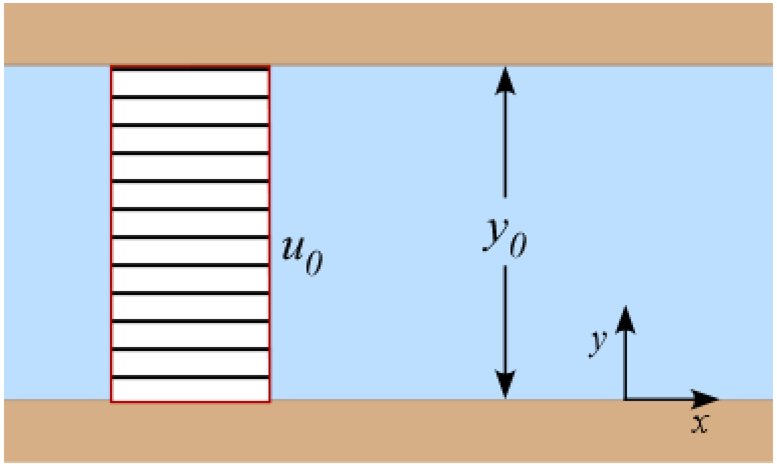
$$\ln\left(\frac{\bar{c}_{max}}{\bar{c}}\right) = \frac{u_0}{4 e_z x}(z - z_{in})^2$$



N.B. The goodness of fitted data depends mainly of the position of z_{in}. When z_{in} is close to $z = z_0, 0$ we can have significant error in the estimation of e_z ($\varepsilon > 10 \div 20\%$).

It is not possible applying the theoretical model of turbulence, that is adopted for the vertical diffusivity, along the transversal direction.

Let's assume a wide open channel. By the Prandtl theory:



$$e_y \propto l^2 \frac{\partial u}{\partial y}$$

In this case:

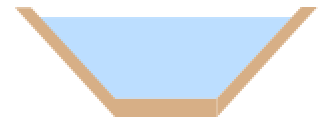
$$\frac{\partial u}{\partial y} = 0 \quad \rightarrow \quad e_y = 0$$

Actually transverse turbulent diffusion exists and it is strictly related to the presence of <u>secondary currents</u> in the open channel.
Thus, the coefficient e_y has to be estimated through experimental analysis!

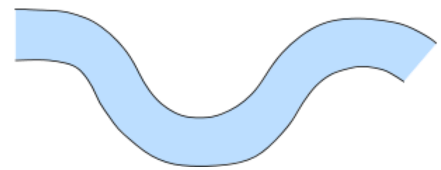
In particular:

i. Wide straight channel: $0.10 < \frac{e_y}{u_* z_0} < 0.26$

ii. Straight prismatic channel: $0.15 < \frac{e_y}{u_* z_0} < 0.30$



iii. Weakly meandering channel: $0.30 < \frac{e_y}{u_* z_0} < 0.90$



The comparison between vertical and transverse diffusion coefficient shows clearly that the latter is the greatest. Indeed:

$$\begin{cases} e_y \cong 0.26\, u_* z_0 \\ e_z = 0.067\, u_* z_0 \end{cases} \Longrightarrow \frac{e_y}{e_z} \cong 4$$

Now let's compare the <u>time scale</u> of the two process.

The time scale of diffusion is: $T \propto \frac{L^2}{D}$

$$\frac{T_y}{T_z} \propto \left(\frac{L_y}{L_z}\right)^2 \frac{e_z}{e_y} = \left(\frac{B}{z_0}\right)^2 \frac{e_z}{e_y} \sim 10^2 \frac{1}{4}$$

L_y is not the vertical mixing length in this case

$$\frac{T_y}{T_z} \sim O(10 \div 10^2)$$

Transverse diffusion can be 1 or 2 orders of magnitude slower than vertical diffusion!

It means that, after the insertion of a lumped solute mass, the process of transverse diffusion occurs when the process of vertical diffusion is ended!

Observations of turbulent flow in rivers show that vortexes have similar length scale along x and z. Thus the diffusivity occurs at the same distance in both the directions.

Moreover, through the force balance of fluid element, it is possible demonstrate that $\tau_{xz} \sim \tau_{xx}$.

Since these two conditions, the diffusivity has the same magnitude along x and z, i.e. $e_x \sim e_z = 0.067 u_* z_0$.

However, the longitudinal diffusion is generally neglected in the studies of solute transport. Indeed, let's consider the 1D flow problem being molecular diffusivity and transverse turbulent diffusion assumed to be zero:

$$\frac{\partial C}{\partial t} + U_0 \frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left(e_x \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial z}\left(e_z \frac{\partial C}{\partial z}\right)$$

All these terms are time-averaged!

To understand the relevance of each term of the equation we use the dimensional analysis. To do this analysis we need of defining one geometric and one kinematic scale quantities of the involved processes.

Geometric quantities:

$$\begin{cases} L_x \rightarrow & \text{Longitudinal length (necessary to observe } C \text{ variation):} \quad x = L_x\,\hat{x} \\ z_0 \rightarrow & \text{Vertical length:} \quad z = z_0\,\hat{z} \end{cases}$$

Kinematic quantities:

$$\begin{cases} U_0 \rightarrow & \text{Longitudinal velocity (related to advection)} \\ u_* \rightarrow & \text{Friction velocity (related to diffusion)} \end{cases}$$

By fixing these scale quantities, we can express the other variables of the equation as following:

$$C = C_0\,\hat{C}$$

$$t = \frac{L_x}{U_0}\,\hat{t}$$

$$e_x = e_z = u_* z_0\,\hat{e}$$

By replacing into the A-D equation:

$$C_0 \frac{U_0}{L_x}\frac{\partial \hat{C}}{\partial \hat{t}} + U_0 \frac{C_0}{L_x}\frac{\partial \hat{C}}{\partial \hat{x}} = \frac{1}{L_x}\frac{\partial}{\partial \hat{x}}\left(u_* z_0 \frac{C_0}{L_x}\hat{e}\frac{\partial \hat{C}}{\partial \hat{x}}\right) + \frac{1}{z_0}\frac{\partial}{\partial \hat{z}}\left(u_* z_0 \frac{C_0}{z_0}\hat{e}\frac{\partial \hat{C}}{\partial \hat{z}}\right)$$

$$\frac{U_0}{L_x}\frac{\partial \hat{C}}{\partial \hat{t}} + \frac{U_0}{L_x}\frac{\partial \hat{C}}{\partial \hat{x}} = \frac{u_* z_0}{{L_x}^2}\frac{\partial}{\partial \hat{x}}\left(\hat{e}\frac{\partial \hat{C}}{\partial \hat{x}}\right) + \frac{u_*}{z_0}\frac{\partial}{\partial \hat{z}}\left(\hat{e}\frac{\partial \hat{C}}{\partial \hat{z}}\right)$$

The group of parameters represents the order of magnitude of each term of equation:

$$\longrightarrow \quad \underbrace{\frac{\partial C}{\partial t} + U_0\frac{\partial C}{\partial x}}_{O\left(\frac{U_0}{L_x}\right)} = \underbrace{\frac{\partial}{\partial x}\left(e_x\frac{\partial C}{\partial x}\right)}_{O\left(\frac{u_* z_0}{{L_x}^2}\right)} + \underbrace{\frac{\partial}{\partial z}\left(e_z\frac{\partial C}{\partial z}\right)}_{O\left(\frac{u_*}{z_0}\right)}$$

Let's compare advection and longitudinal diffusion terms:

$$\frac{\dfrac{\partial C}{\partial t}+U_0\dfrac{\partial C}{\partial x}}{\dfrac{\partial}{\partial x}\left(e_x\dfrac{\partial C}{\partial x}\right)} \propto \frac{U_0}{L_x}\frac{{L_x}^2}{u_* z_0}=\frac{U_0}{u_*}\frac{L_x}{z_0}\cong 10\cdot\frac{10^2 \div 10^3}{1 \div 10} \sim O(10^2 \div 10^3) \gg 1$$

Longitudinal diffusion is negligible!

Let's compare advection and vertical diffusion terms:

$$\frac{\dfrac{\partial C}{\partial t}+U_0\dfrac{\partial C}{\partial x}}{\dfrac{\partial}{\partial z}\left(e_z\dfrac{\partial C}{\partial z}\right)} \propto \frac{U_0}{L_x}\frac{z_0}{u_*}=\frac{U_0}{u_*}\frac{z_0}{L_x}\cong 10\cdot\frac{1 \div 10}{10^2 \div 10^3} \sim O(1 \div 10^{-1}) \sim 1$$

Vertical diffusion and advection are comparable!

Chezy expression: $U_0=\chi\sqrt{z_0 i_b}$

$u_*=\sqrt{\tau_0/\rho}$

$$\longrightarrow \quad \frac{U_0}{u_*}=\frac{\chi\sqrt{z_0 i_b}}{\sqrt{\tau_0/\rho}}=\frac{\chi\sqrt{z_0 i_b}}{\sqrt{g z_0 i_b}}=\frac{\chi}{\sqrt{g}}\cong\frac{30 \div 45}{3}\sim 10$$