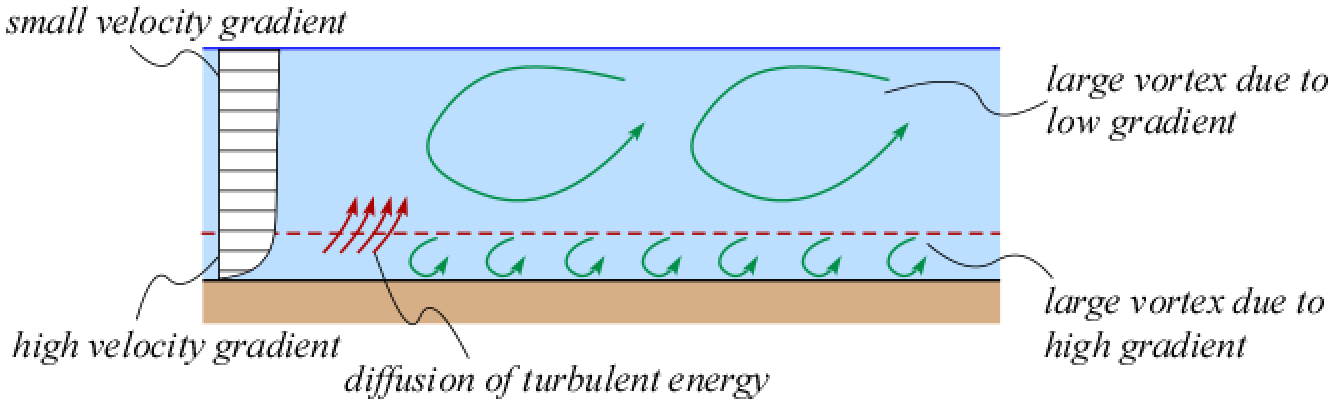
LESSON 6: TURBULENCE AND TURBULENT DIFFUSION

ETHIOLOGY

Turbulence is a random process due to the vorticity of velocity field. It rises in presence of velocity gradient, that causes the rising of large vortexes.



The vortexes within the boundary layer have higher intensity and stronger anisotropy than the vortexes in the external region.

Vortex is defined by the rotor of the velocity:

$$\nabla \times \boldsymbol{u} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{bmatrix} = \begin{bmatrix} \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} - \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{bmatrix}$$

Macrovortex
- Anisotropic-etherogeneous
- Energy from mean flow

$\longrightarrow$

Direct Energy cascade

Microvortex
- Isotropic-homogeneous
- Dissipation due to viscosity

Turbulence characteristics:

i. Turbulence is highly irregular

ii. Velocity fluctuations are random in t and $\boldsymbol{x}$

iii. Variables of interest can be defined as $\boldsymbol{a} = < \boldsymbol{a} > + \boldsymbol{a}'$

iv. The process is dissipative

v. The flow is rotational

vi. The process is 3D

vii. It is anisotropic for the large length-scale

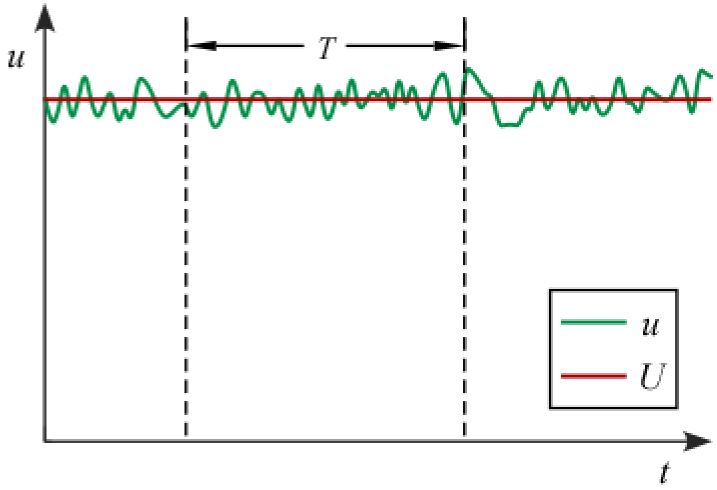
VELOCITY DECOMPOSITION

In presence of turbulent velocity field, the velocity $\boldsymbol{u}$ can be expressed as proposed by Reynolds:

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$$

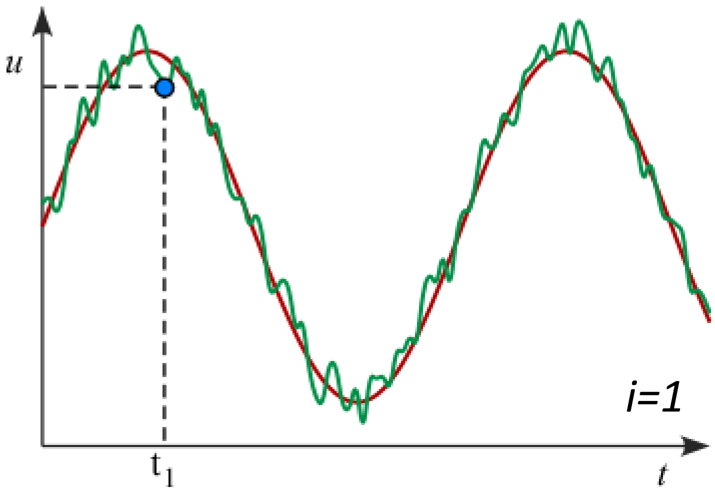
With: $\boldsymbol{U} = <\boldsymbol{u}>$ mean velocity ($<\cdot>$ time averaged)

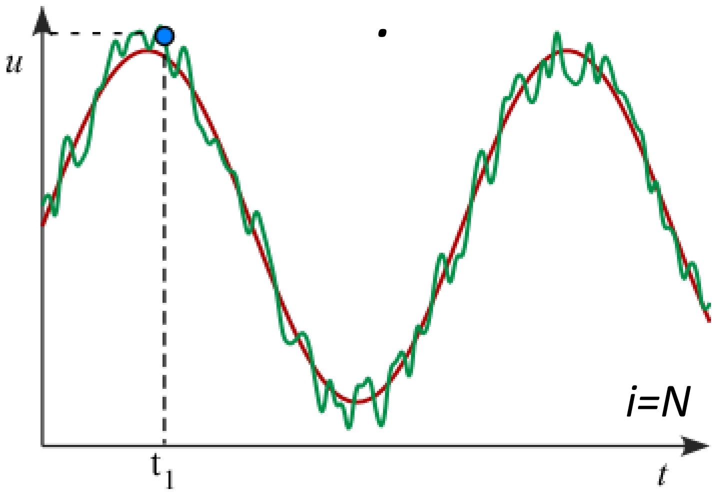
$\boldsymbol{u}'$ velocity fluctuation ($<\boldsymbol{u}'> = 0$)



Steady flow

$$\boldsymbol{U} = \frac{1}{T}\int_0^T \boldsymbol{u}\,\mathrm{d}t$$





Unsteady flow

$$\boldsymbol{U} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{u_i}$$

Hardly feasible in environmental systems

Probabilistic approach

$$< \boldsymbol{u}(\boldsymbol{x},t) >= \int_{-\infty}^{+\infty} \boldsymbol{u}(\boldsymbol{x},t)\, \underbrace{\mathrm{p}(\boldsymbol{u}|\boldsymbol{x},t)}_{\text{Probabilistic density function}}\, \mathrm{d}\boldsymbol{u}$$

TURBULENT DIFFUSION EQUATION

Let’s start from the A-D equation replacing $\boldsymbol{u}$ and c with their mean and fluctuation:

$$\frac{\partial (C + c')}{\partial t} + (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla (C + c') = D\, \nabla^2 (C + c')$$

This equation can be simplified averaging in time:

$$< \frac{\partial (C + c')}{\partial t} + (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla (C + c') = D\, \nabla^2 (C + c') >$$

$$< a + b > = < a > + < b >$$

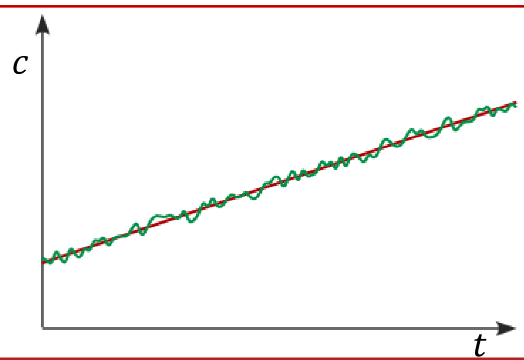
Being average a linear operator:

$$\frac{1}{T}\int_0^T (a+b)\mathrm{d}t = \frac{1}{T}\int_0^T a\,\mathrm{d}t + \frac{1}{T}\int_0^T b\,\mathrm{d}t$$

$$< \frac{\partial (C + c')}{\partial t} > + < (\boldsymbol{U} + \boldsymbol{u}') \cdot \nabla (C + c') > = < D\, \nabla^2 (C + c') >$$

a b c

Similarly to $\mathbf{u}$:



$$c = < c > + c' = C + c'$$

Let's analyze each term:

a

$$< \frac{\partial (C + c')}{\partial t} > = \frac{\partial < (C + c') >}{\partial t}$$

Derivative is linear!

$$\frac{\partial < (C + c') >}{\partial t} = \frac{\partial < C >}{\partial t} + \underbrace{\cancel{\frac{\partial < c' >}{\partial t}}}_{0}$$

$$\longrightarrow \quad < \frac{\partial (C + c')}{\partial t} > = \frac{\partial C}{\partial t}$$

c

$$< D\, \nabla^2 (C + c') > = D\, \nabla^2 < (C + c') >$$

Laplacian is linear!

$$D\, \nabla^2 < (C + c') > = D\, \nabla^2 < C > + D\, \nabla^2 < \underbrace{\cancel{c'}}_{0} >$$

$$\longrightarrow \quad < D\, \nabla^2 (C + c') > = D\, \nabla^2 C$$

b $<(\boldsymbol{U}+\boldsymbol{u}')\cdot\nabla(C+c')>$ *It is not linear! Turbulent diffusion rises from this not linearity.*

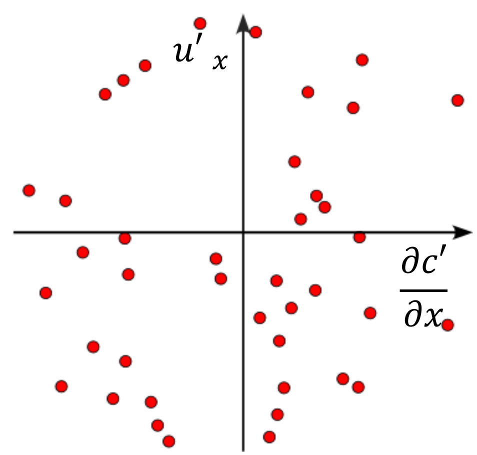
$$\underbrace{<\boldsymbol{U}\cdot\nabla C>}_{1}+\underbrace{<\boldsymbol{U}\cdot\nabla c'>}_{2}+\underbrace{<\boldsymbol{u}'\cdot\nabla C'>}_{3}+\underbrace{<\boldsymbol{u}'\cdot\nabla c'>}_{4}$$

1 $<\boldsymbol{U}\cdot\nabla C> = <\boldsymbol{U}>\cdot<\nabla C> = <\boldsymbol{U}>\cdot\nabla<C> = \boldsymbol{U}\cdot\nabla C$ *Gradient is linear!.*

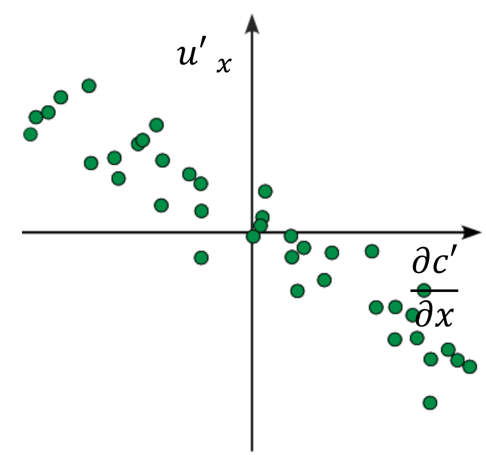
2 $<\boldsymbol{U}\cdot\nabla c'> = <\boldsymbol{U}>\cdot<\nabla c'> = <\boldsymbol{U}>\cdot\nabla<\cancelto{0}{c'}> = 0$

3 $<\boldsymbol{u}'\cdot\nabla C> = <\boldsymbol{u}'>\cdot<\nabla C> = <\cancelto{0}{\boldsymbol{u}'}>\cdot\nabla<C> = 0$

4 $<\boldsymbol{u}'\cdot\nabla c'> = ?$



$<\boldsymbol{u}'\cdot\nabla c'> = 0$



$\boxed{<\boldsymbol{u}'\cdot\nabla c'> \neq 0}$

The time averaged A-D equation that takes into account turbulence reads:

$$\frac{\partial C}{\partial t} + \boldsymbol{U} \cdot \nabla C = \; D\,\nabla^2 C - < \boldsymbol{u}' \cdot \nabla c' >$$

Additional term of diffusivity due to turbulent fluctuation

N.B. Additional term appears similarly to the one in the Reynolds momentum equation which represents the apparent viscosity.

Even though the problem has been simplified by averaging the equation, there are two more variables in the previous equation ($\boldsymbol{u}'$ and c', i.e. 4 unknown parameters).

We have to reduce the problem!

Reynolds equation: $\nabla(P + \gamma h) = -\rho \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} + \mu\,\nabla^2 \boldsymbol{U} - \rho < \boldsymbol{u}' \cdot \nabla \boldsymbol{u}' >$

where: $p = P + p'$ *is the fluid pressure (P is the averaged pressure)*

γh *is the gravity force*

CONTINUITY EQUATION

The continuity equation can be developed in terms of mean velocity and turbulent fluctuations, as following :

$$\nabla \cdot \boldsymbol{u} = \nabla \cdot (\boldsymbol{U} + \boldsymbol{u}') = \nabla \cdot \boldsymbol{U} + \nabla \cdot \boldsymbol{u}' = 0$$

Also in this case we can average the equation, that reads:

$$< \nabla \cdot (\boldsymbol{U} + \boldsymbol{u}') > = < \nabla \cdot \boldsymbol{U} > + < \nabla \cdot \boldsymbol{u}' > = 0$$

$$< \nabla \cdot \boldsymbol{U} > + < \nabla \cdot \boldsymbol{u}' > = \nabla \cdot < \boldsymbol{U} > + \nabla \cdot \underbrace{< \boldsymbol{u}' >}_{0} = 0$$

It means: $\nabla \cdot \boldsymbol{U} = 0$

And then, for the original continuity equation: $\nabla \cdot \boldsymbol{u}' = 0$

$$\begin{cases} \nabla \cdot \boldsymbol{U} = 0 \\ \nabla \cdot \boldsymbol{u}' = 0 \end{cases}$$

<u>Continuity Equation for turbulent uncompressible fluid</u>

The definition of the continuity allows us to rearrange the additional term of diffusivity, as follows:

$$< \boldsymbol{u}' \cdot \nabla c' > = < \boldsymbol{u}' \cdot \nabla c' + c' \nabla \cdot \boldsymbol{u}' >$$

$$< \boldsymbol{u}' \cdot \nabla c' + c' \nabla \cdot \boldsymbol{u}' > = < \nabla \cdot (c' \, \boldsymbol{u}') > = \nabla \cdot < c' \, \boldsymbol{u}' >$$

To solve the problem we should find a function *f*, such that: $f(C, \boldsymbol{U}) = \nabla \cdot < c' \, \boldsymbol{u}' >$

Let's define the mass turbulent flux as:

$$\boldsymbol{q}^t = < c' \, \boldsymbol{u}' > = < (c - C)(\boldsymbol{u} - \boldsymbol{U}) >$$

$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{U}$ *is similar to* $\boldsymbol{v}_s = \boldsymbol{u}_s - \boldsymbol{U}$ *(see Lesson 2)*

By Taylor's hypothesis, there is analogy between $\boldsymbol{q}^t$ and Fick's law:

$$\rightarrow \begin{cases} \boldsymbol{q}^t = -\boldsymbol{E} \, \nabla C & \text{Turbulent Diffusion} \\ \boldsymbol{q}^r = -D \, \nabla C & \text{Molecular Diffusion} \end{cases}$$

$\boldsymbol{E}$ is the Tensor of Turbulent Diffusion. It is defined as following:

$$\boldsymbol{E} = \begin{bmatrix} e_{xx} & e_{yx} & e_{zx} \\ e_{xy} & e_{yy} & e_{zy} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix} \xrightarrow{\text{principal axes}} \boldsymbol{E} = \begin{bmatrix} e_x & 0 & 0 \\ 0 & e_y & 0 \\ 0 & 0 & e_z \end{bmatrix}$$

The A-D equation now reads:

$$\longrightarrow \quad \frac{\partial C}{\partial t} + \boldsymbol{U} \cdot \nabla C = D\, \nabla^2 C + \nabla \cdot (\boldsymbol{E}\, \nabla C) \qquad \text{Turbulent A-D equation}$$

In natural systems it is more effective assuming the principal axes as reference system of the problem and rearraging the A-D equation as following:

$$\longrightarrow \quad \frac{\partial C}{\partial t} + \boldsymbol{U} \cdot \nabla C = D\, \nabla^2 C + \frac{\partial}{\partial x}\left(e_x \frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(e_y \frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(e_z \frac{\partial C}{\partial z}\right)$$

N.B. In our cases of study: $D \ll \min\left(e_x, e_y, e_z\right)$

To quantify the flow turbulence, we define:

$$e_c = \rho \frac{< u'^2 + v'^2 + w'^2 >}{2} = \rho \frac{< {u'_i}^2 >}{2}$$

Turbulent Kinetic Energy

$$\sqrt{< {u'_i}^2 >} = \sqrt{\frac{u'^2 + v'^2 + w'^2}{3}} = \sigma_u$$

Turbulent Intensity

Here σ_u is the standard deviation of the velocity fluctuations (${\sigma_u}^2$ is the fluctuation variance).

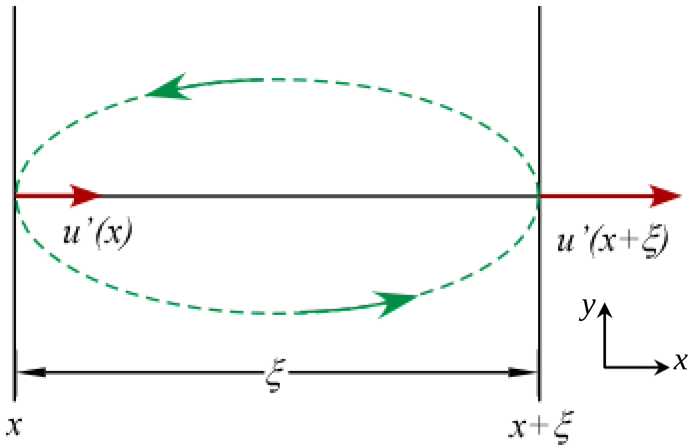
$$\frac{\sigma_u}{< u_i >} = \frac{\sqrt{< {u'_i}^2 >}}{|\boldsymbol{U}|}$$

$\boldsymbol{u} = < \boldsymbol{u} > + \boldsymbol{u}'$
$< \boldsymbol{u} > = \boldsymbol{U}$

Relative Turbulent Intensity

N.B. these parameters allow us to quantify turbulence, but they do not explain the vortexes distribution

To know the vortexes influence, it is useful introducing the correlation coefficients (they are functions of t and $\boldsymbol{x}$).



The vortex affects the magnitude of the velocity fluctuation

the correlation coefficients is then:

Correlation Function

$$f(\xi) = \frac{< u'(x)u'(x+\xi) >}{\sqrt{< u'^2(x) >}\,\sqrt{< u'^2(x+\xi) >}}$$

It normalizes the correlation function

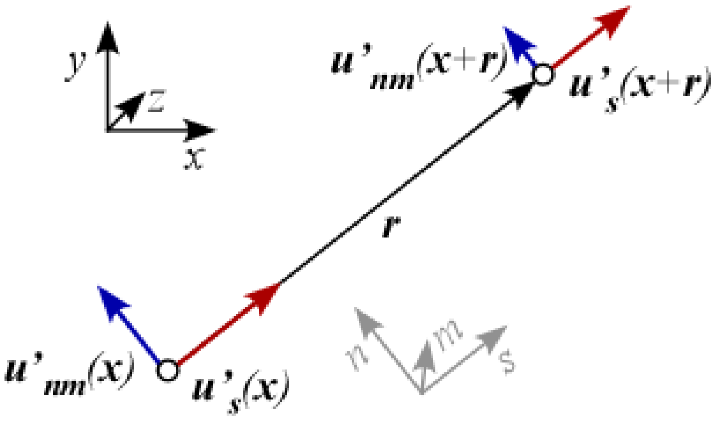
In the 3D case:

Longitudinal correlation coefficient

$$f(\boldsymbol{x},\boldsymbol{r},t)=\frac{<\boldsymbol{u'}_{\boldsymbol{s}}(\boldsymbol{x},t)\boldsymbol{u'}_{\boldsymbol{s}}(\boldsymbol{x}+\boldsymbol{r},t)>}{\sqrt{<\boldsymbol{u'}_{\boldsymbol{s}}{}^2(\boldsymbol{x},t)>}\sqrt{<\boldsymbol{u'}_{\boldsymbol{s}}{}^2(\boldsymbol{x}+\boldsymbol{r},t)>}}$$

Transverse correlation coefficient

$$g(\boldsymbol{x},\boldsymbol{r},t)=\frac{<\boldsymbol{u'}_{\boldsymbol{nm}}(\boldsymbol{x},t)\boldsymbol{u'}_{\boldsymbol{nm}}(\boldsymbol{x}+\boldsymbol{r},t)>}{\sqrt{<\boldsymbol{u'}_{\boldsymbol{nm}}{}^2(\boldsymbol{x},t)>}\sqrt{<\boldsymbol{u'}_{\boldsymbol{nm}}{}^2(\boldsymbol{x}+\boldsymbol{r},t)>}}$$



These two coefficients help us to understand the spatial structure of vortexes. To understand the time evolution of the vortexes we define the following coefficient:

$$R_E(\boldsymbol{x},t,\tau)=\frac{<\boldsymbol{u}'(\boldsymbol{x},t)\boldsymbol{u}'(\boldsymbol{x},t+\tau)>}{\sqrt{<\boldsymbol{u}'^2(\boldsymbol{x},t)>}\sqrt{<\boldsymbol{u}'^2(\boldsymbol{x},t+\tau)>}}$$

Eulerian temporal correlation coefficient

ERGODIC PROCESS

Process is **ergodic** if it is stationary and homogenous.

Stationary turbolence means that averaged quantities do not depend on time t.

→ To mantain stationarity, it is necessary a constant production of turbulence

Homogeneous turbulence means that averaged quanties do not depend on space $\boldsymbol{x}$.

→ To observe homogeneity, the domain should be extended indefinetely

Characteristics of ergodicity:

i. All types of average converge to the same value.
ii. The probability density function (pdf) of averaged variables is Gaussian (for the Central Limit Theorem).
iii. The ergodic variables become statistically independent and no-correlate

Under ergodicity, correlation coefficients do not depend on $\boldsymbol{x}$ or t, but on their difference

$f(\boldsymbol{x}, \boldsymbol{r}, t) \longrightarrow f(r, t) = f(r)$

$g(\boldsymbol{x}, \boldsymbol{r}, t) \longrightarrow g(r, t) = g(r)$

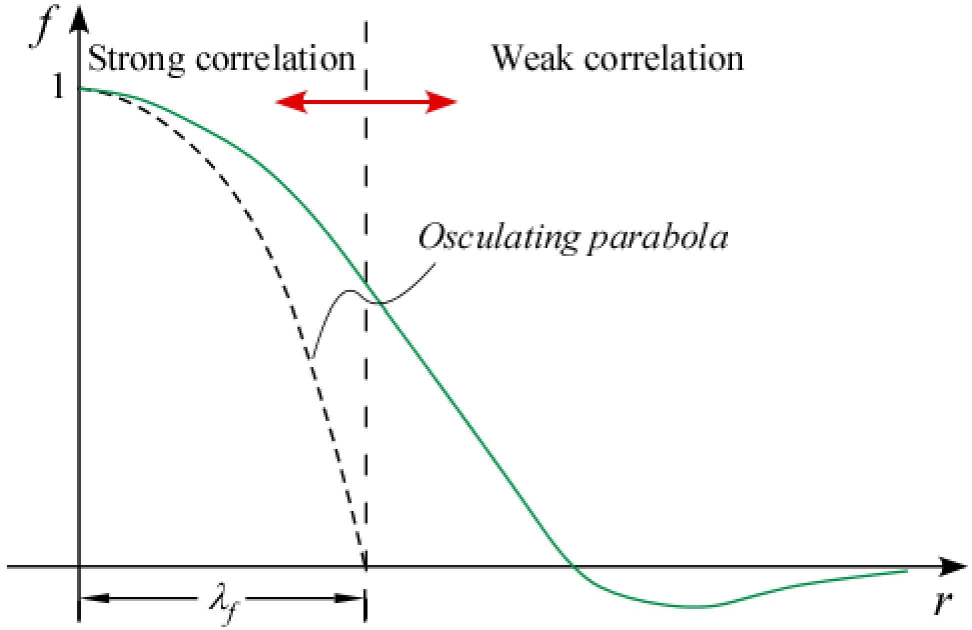
Usually we fixed t.

$R_E(\boldsymbol{x}, t, \tau) \longrightarrow R_E(\boldsymbol{x}, \tau) = R_E(\tau)$ *Usually we fixed $\boldsymbol{x}$.*

DEFINITION OF TURBULENCE LENGTH SCALES

Let's analyze $f(r)$ (same consideration can be done for the other coefficients). The correlation function has the following properties:

$$\begin{cases} f(0) = 1 \\ f(r) = f(-r) \\ f(r) \leq 1 \end{cases}$$



λ_f is the length that defines the longitudinal size of microvortexes. It is determined by Taylor series expansion of $f(r)$

$$f(r) \cong \cancelto{1}{f(0)} + r \cancelto{0}{\left.\frac{\mathrm{d}f}{\mathrm{d}r}\right|_{r=0}} + \frac{r^2}{2} \left.\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\right|_{r=0} + O(r^4)$$

Equation of Osculating Parabola

$$f(r) \cong 1 - \frac{r^2}{2}\frac{2}{\lambda_f{}^2} \longrightarrow f(r) \cong 1 - \frac{r^2}{\lambda_f{}^2}$$

Finally we can determined:

$$\begin{cases} \dfrac{1}{{\lambda_f}^2} = -\dfrac{1}{2}\dfrac{\mathrm{d}^2 f}{\mathrm{d}r^2}\bigg|_{r=0} \\ \Lambda_f = \displaystyle\int_0^\infty f(r)\,\mathrm{d}r \end{cases}$$

$\longrightarrow$ Longitudinal length scale of microvortexes

$\longrightarrow$ Longitudinal length scale of macrovortexes

$$\begin{cases} \dfrac{1}{{\lambda_g}^2} = -\dfrac{1}{2}\dfrac{\mathrm{d}^2 g}{\mathrm{d}r^2}\bigg|_{r=0} \\ \Lambda_g = \displaystyle\int_0^\infty g(r)\,\mathrm{d}r \end{cases}$$

$\longrightarrow$ Transversal length scale of microvortexes

$\longrightarrow$ Transversal length scale of macrovortexes

$$\begin{cases} \dfrac{1}{{\tau_E}^2} = -\dfrac{1}{2}\dfrac{\mathrm{d}^2 R_E}{\mathrm{d}\tau^2}\bigg|_{\tau=0} \\ \mathrm{T}_E = \displaystyle\int_0^\infty R_E(\tau)\,\mathrm{d}\tau \end{cases}$$

$\longrightarrow$ Time scale of microvortexes

$\longrightarrow$ Time scale of macrovortexes

APPROXIMATION OF FROZEN TURBULENCE

It is based on the Taylor's hypothesys, i.e.:

i. 1D flow, i.e. $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$ $\boldsymbol{U} = (U, 0, 0)$ $\boldsymbol{u}' = (u', v', w')$

ii. Turbulence is stationary and homogenous, i.e the process is ergodic

Under these assuptions the Navier-Stokes equation along x becomes :

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \, \nabla^2 \boldsymbol{u}$$

$$\cancelto{0}{\frac{\partial U}{\partial t}} + \frac{\partial u'}{\partial t} + U \cancelto{0}{\frac{\partial U}{\partial x}} + U \frac{\partial u'}{\partial x} + u' \cancelto{0}{\frac{\partial U}{\partial x}} + u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} + w' \frac{\partial u'}{\partial z}$$

$$= -\frac{1}{\rho} \left(\cancelto{0}{\frac{\partial P}{\partial x}} + \frac{\partial p'}{\partial x} \right) + \nu \cancelto{0}{\frac{\partial^2 U}{\partial x^2}} + \nu \left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2} \right)$$

APPROXIMATION OF FROZEN TURBULENCE

We can understand the role of each term through the dimensional analysis of the equation

$U \propto U_0$ → Velocity scale of mean flow

$u', v', w' \propto u_0$ → Velocity scale of fluctuations

$x, y, z \propto L_0$ → Length scale of the problem

$$
\begin{cases}
\dfrac{\partial u'}{\partial t} & \propto \quad \dfrac{u_0 U_0}{L_0} \quad \dfrac{L_0}{{u_0}^2} = \dfrac{U_0}{u_0} \\
U \dfrac{\partial u'}{\partial x} & \propto \quad \dfrac{u_0 U_0}{L_0} \quad \dfrac{L_0}{{u_0}^2} = \dfrac{U_0}{u_0} \\
u' \dfrac{\partial u'}{\partial x} & \propto \quad \dfrac{{u_0}^2}{L_0} \quad \dfrac{L_0}{{u_0}^2} = 1 \\
\dfrac{1}{\rho} \dfrac{\partial p'}{\partial x} & \propto \quad \dfrac{1}{\rho} \dfrac{\rho {u_0}^2}{L_0} \quad \dfrac{L_0}{{u_0}^2} = 1 \\
\nu \dfrac{\partial^2 u'}{\partial x^2} & \propto \quad \nu \dfrac{u_0}{{L_0}^2} \quad \dfrac{L_0}{{u_0}^2} = \dfrac{1}{Re}
\end{cases}
$$

Dimensionless N-S equation is then:

$$\frac{U_0}{u_0}\frac{\partial \hat{u}'}{\partial \hat{t}} + \frac{U_0}{u_0}\frac{\partial \hat{u}'}{\partial \hat{x}} = -\left(\hat{u}'\frac{\partial \hat{u}'}{\partial \hat{x}} + \hat{v}'\frac{\partial \hat{u}'}{\partial \hat{y}} + \hat{w}'\frac{\partial \hat{u}'}{\partial \hat{z}}\right) - \frac{\partial \hat{p}'}{\partial \hat{x}} + \frac{1}{Re}\nabla^2 \hat{\boldsymbol{u}}'$$

Given:

- $U \gg |\boldsymbol{u}'|$
- $Re \gg 1$

$$\longrightarrow \frac{U_0}{u_0}\frac{\partial \hat{u}'}{\partial \hat{t}} + \frac{U_0}{u_0}\frac{\partial \hat{u}'}{\partial \hat{x}} = 0 \longrightarrow \frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} = 0$$

Finally:

$$\longrightarrow \frac{\partial}{\partial t} \approx -U\frac{\partial}{\partial x}$$

<u>Frozen turbulence</u>

Length scale and temporal scale are linked by the mean velocity U

From this equivalence: $\tau = \frac{r}{U}$

$$\longrightarrow \begin{aligned} f(r) &= R_E(\tau) \\ \lambda_f &= U\tau_E \\ \Lambda_f &= U\mathrm{T}_E \end{aligned}$$

REPRESENTATION OF FROZEN TURBULENCE