Methods and Models for Combinatorial Optimization

Modeling by Linear Programming

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1 Linear programming models

Linear programming models are a special class of *mathematical programming models*. A mathematical programming model is used to describe the characteristics of the optimal solution of an optimization problem by means of mathematical relations. Besides giving a formal description of the problem, the model constitutes the basis for the application of standard optimization algorithms (available as algebraic modeling systems and optimization software) capable of finding an optimal solution. In the following we review the elements of a mathematical programming model; as an example, we will refer to the following simple optimization problem:

Example 1 *A perfume firm produces two new items by mixing three essences: rose, lily and violet. For each decaliter of perfume* one*, it is necessary to use 1.5 liters of rose, 1 liter of lily and 0.3 liters of violet. For each decaliter of perfume* two*, it is necessary to use 1 liter of rose, 1 liter of lily and 0.5 liters of violet. 27, 21 and 9 liters of rose, lily and violet (respectively) are available in stock. The company makes a profit of 130 euros for each decaliter of perfume* one *sold, and a profit of 100 euros for each decaliter of perfume* two *sold. The problem is to determine the optimal amount of the two perfumes that should be produced.*

A mathematical programming model consists of the following elements.

- **Sets**, which group the elements of the system. In the example, two sets arise naturally: the essence set ($I = \{rose, lily, violet\}$) and the perfume set ($J = \{one, two\}$).
- **Parameters**, i.e, the data of the problem, which represent the known quantities depending on the elements of the system. The following parameters appear in the example: the profits, defined for each perfume (130 euros for each decaliter of perfume *one* and 100 euros for each decaliter of perfume *two*), the stock availability, defined for each essence (27, 21, 9 liters for *rose*, *lily* and *violet*, respectively), and the amount of essences needed for each perfume, defined for each pair essence-perfume (for instance, 1 liter of lily for each decaliter of perfume *one*, 0.5 liter of violet for each decaliter of perfume *two*, and so forth).

- **Decision (or control) variables**: these are the unknown quantities, on which we can act in order to find different possible solutions to the problem. In the example, the decision variables are the amount (in decaliters) of each perfume to be produced (we denote these variables by x_{one} and x_{two}).
- **Constraints**: these are the mathematical relations that describe conditions imposing the feasibility of the solutions. In other words, the constraints distinguish between the combinations of values of the variables representing acceptable solutions and the combinations of values giving non-acceptable solutions. For instance, no feasible solution can use more than 27 liters of essence *rose*, which can be written as $1.5x_{one} + 1x_{two} \leq 27$.
- **Objective function**: this is the quantity to maximize or minimize, written as a function of the decision variables. In the example we want to maximize the profit in euros, i.e., the expression $130x_{one} + 100x_{two}$.

Solving an optimization problem formulated as a mathematical programming model means deciding the values of the variables that satisfy all the constraints and maximize or minimize the objective function. These values are the *solution* to the problem.

A *Linear Programming model* is a mathematical programming model in which

- the objective function is a *linear* expression of the decision variables;
- the constraints are given by a system of *linear* equations and/or inequalities.

Depending on the nature of the *domain* of the decision variables, one has:

- Linear Programming models (strictly speaking; abbreviated as LP) if all the variables can take real values;
- Integer Linear Programming models (ILP) if all the variables are allowed to take integer values only;
- Mixed Integer Linear Programming models (MILP) if some variables can take real values and others are allowed to take integer values only.

The problem given in the example can therefore be formulated as a Linear Programming problem:

$$
\begin{array}{rrcrcrl}
\max & 130x_{one} & + & 100x_{two} & & & \text{objective function} \\
s.t. & 1,5x_{one} & + & x_{two} & \leq & 27 & \text{availability of rose} \\
 & x_{one} & + & x_{two} & \leq & 21 & \text{availability of lily} \\
 & 0,3x_{one} & + & 0,5x_{two} & \leq & 9 & \text{availability of violet} \\
 & x_{one} & , & x_{two} & \geq & 0 & \text{domains of the variables}
\end{array}
$$

It is convenient to describe the models in a more general form, by exploiting suitable algebraic notation, so that the same model encompasses the description of many cases (or *instances*) of the same *problem*, where "problem" means "class of problems defined in the same way". To this purpose, once the sets and the corresponding indices have been defined, one can generalize the definition of parameters and variables, and thus of the objective function and constraints. In the example, we could define the problem of optimizing production, where one has to determine how much to produce for each item in order to maximize profit, under the condition that resource availability is not exceeded. By using the following notation:

- I: resource set (in the example, essences);
- J: product (or item) set (in the example, perfumes);

- D_i: availability of resource $i \in I$;
- P_j: unit profit for each product $j \in J$;
- Q_{ij}: amount of resource $i \in I$ required for each unit of product $j \in J$;

- x_j: amount of product $j \in J$ (decision variables)

the model for the (general) problem can be written as follows:

$$\begin{array}{rlll} \max & \sum_{j \in J} P_j x_j & & \\ s.t. & \sum_{j \in J} Q_{ij} x_j \leq D_i & \forall & i \in I \\ & x_j \in \mathbb{R}_+ & \forall & j \in J \end{array}$$

The instance described in the example is obtained by setting $I = \{rose, lily, violet\}$, $J = \{one, two\}$ and fixing the values of the parameters D_i, P_j e Q_{ij}. By choosing different values for the sets and/or the parameters, one obtains different instances of the same problem, for which the model is still valid.

Remark. The general model described above coincides with the basic modeling scheme for the problem of finding the optimal production mix; the perfume problem is an instance of this general problem.

2 Exercises

Give a linear programming formulation for the following optimization problems. For each of them, provide a general formulation capable of modeling different instances of the same problem.

1. A construction company has to move the scaffolds from three closing building sites (A, B, C) to three new building sites (1, 2, 3). The scaffolds consist of iron rods: in the sites A, B, C there are respectively 7000, 6000 and 4000 iron rods, while the new sites 1, 2, 3 need 8000, 5000 and 4000 rods respectively. The following table provide the cost of moving one iron rod from a closing site to a new site:

Costs (euro cents)	**1**	**2**	**3**
A	9	6	5
B	7	4	9
C	4	6	3

   Trucks can be used to move the iron rods from one site to another site. Each truck can carry up to 10000 rods. Find a linear programming model that determine the minimum cost transportation plan, taking into account that:

   - using a truck causes an additional cost of 50 euros;
   - only 4 trucks are available (and each of them can be used only for a single pair of closing site and new site);
   - the rods arriving in site 2 cannot come from both sites A and B;
   - it is possible to rent a fifth truck for 65 euros (i.e., 15 euros more than the other trucks).

     Hint: start from the basic modeling scheme for the transportation problem and from the classical modeling scheme for fixed costs that uses *big-M* constraints.

2. A telephone company wants to install antennas in some sites in order to cover six areas. Five possible sites for the antennas have been detected. After some simulations, the intensity of the signal coming from an antenna placed in each site has been established for each area. The following table summarized these intensity levels:

	area 1	**area 2**	**area 3**	**area 4**	**area 5**	**area 6**
site A	10	20	16	25	0	10
site B	0	12	18	23	11	6
site C	21	8	5	6	23	19
site D	16	15	15	8	14	18
site E	21	13	13	17	18	22

Receivers recognize only signals whose level is at least 18. Furthermore, it is not possible to have more than two signals reaching level 18 in the same area, otherwise this would cause an interference. Finally, an antenna can be placed in site E only if an antenna is installed also in site D (this antenna would act as a bridge). The company wants to determine where antennas should be placed in order to cover the maximum number of areas.

Hint: this model can be obtained starting from the basic scheme for coverage problems.

3. Constructing a boat requires the completion of the following operations (the table also gives the number of days needed for each operation):

Operations	**Duration**	**Precedences**
A	2	none
B	4	A
C	2	A
D	5	A
E	3	B,C
F	3	E
G	2	E
H	7	D,E,G
I	4	F,G

Some of the operations are alternative to each other. In particular, only one of B and C needs to be executed, and only one of F and G needs to be executed. Furthermore, if both C and G are executed, the duration of I increases by 2 days. The table also shows the precedences for each operation (i.e., operations that must be completed before the beginning of the new operation). For instance, H can start only after the completion of E, D and G (if G will be executed). Write a linear programming model that can be used to decide which operations should be executed in order to minimize the total duration of the construction of the boat.

4. Four Italian friends (Andrea, Bruno, Carlo and Dario) share an apartment. Every Saturday they receive four newspapers: “La Repubblica”, “Il Messaggero”, “La Stampa” and “La Gazzetta dello Sport”, which they read before going out. As they are very nit-picking, each of them wants to read all newspapers in a specific order. Andrea wants to start with “La Repubblica” for one hour, then he reads “La Stampa” for 30 minutes, he only has a look at “Il Messaggero” for two minutes and then finishes with “La Gazzetta dello Sport” for 5 minutes. Bruno prefers to start with “La Stampa”, which he reads for 75 minutes; he then has a look at “Il Messaggero” for three minutes, then he reads “La Repubblica” for 25 minutes and finally “La Gazzetta dello Sport” for 10 minutes. Carlo starts with “Il Messaggero” for 5 minutes, then he reads “La Stampa” for 15 minutes, “La Repubblica” for 10 minutes and “La Gazzetta dello Sport” for 30 minutes. Finally, Dario starts with “La Gazzetta dello Sport” for 90 minutes and then dedicates just one minute to each of “La Repubblica”, “La Stampa” and “Il Messaggero” in this order. Each of the four friends is so insistent with his request of reading the newspapers in his preferred order, that he is willing to wait and read nothing until the newspaper that he wants becomes available. Moreover, none of them would stop reading a newspaper and resume later. By taking into account that Andrea gets up at 8:30, Bruno and Carlo at 8:45 and Dario at 9:30, and that they can wash, get dressed and have breakfast while reading the newspapers, write a linear programming formulation that can be used to decide when each of them should start to read each newspaper, so that they can leave home together as soon as possible.

5. The pharmacy federation wants to organize the opening shifts on public holy days all over the region. The number of shifts is already decided, and the number of pharmacies open on the same day has to be as balanced as possible. Furthermore, every pharmacy is part of one shift only. For instance, if there are 12 pharmacies and the number of shifts is 3, every shift will consist of 4 pharmacies. Pharmacies and users are thought as concentrated in centroids (for instance, villages). For every centroid, the number of users and pharmacies are known. The distance between every ordered pair of centroids is also known. For the sake of simplicity, we ignore congestion problems and we assume that every user will go to the closest open pharmacy. The target is to determine the sifts so that the total distance covered by the users is minimized.

6. *(Distributing energy)* A company distributing electric energy has several power and distributing stations connected by wires. Each station i can:

   - produce p_i kW of energy ($p_i = 0$ if the station cannot produce energy);
   - distribute energy on a sub-network whose users have a total demand of d_i kW ($d_i = 0$ if the station serve no users);
   - carry energy from/to different stations.

   The wires connecting station i to station j have a maximum capacity of u_{ij} kW and a cost of c_{ij} euros for each kW carried by the wires. The company wants to determine the minimum cost distribution plan, under the assumption that the total amount of energy produced equals the total amount of energy required by all sub-networks.

7. *(Distributing different kinds of energy)* A company distributing electric energy has several power and distributing stations connected by wires. Each station produces/distributes different kinds of energy. Each station i can:

   - produce p_i^k kW of energy of type k ($p_i^k = 0$ if the station does not produce energy of type k);
   - distribute energy of type k on a sub-network whose users have a total demand of d_i^k kW ($d_i^k = 0$ if the station serve no users requiring energy of type k);
   - carry energy from/to different stations.

   Note that every station can produce and/or distribute different types of energy. The wires connecting station i to station j have a maximum capacity of u_{ij} kW, independently of the type of energy carried. The transportation cost depends both on the pair of stations (i, j) and the type of energy k, and is equal to c_{ij}^k euros for each kW. The company wants to determine the minimum cost distribution plan, under the assumption that, for each type of energy, the total amount produced equals the total amount of energy of the same type required by all sub-networks.

8. A communication network consists of routers and connections routes between pairs of routers. Every router generates traffic towards every other router and, for every (ordered) pair of routers, the traffic demand has been estimated (this demand is measured in terms of bandwidth required). The traffic from router i to router j uses *multi-hop* technology (the traffic is allowed to go through intermediate nodes) and *splittable flow* technology (the traffic can be split along different paths). For every route, the capacity (how much flow can be sent) is known, and the unit cost for each unit of flow is also known. The target is to send the data flow at the minimum cost.

2.1 Solution of Exercise 1

We give a possible general formulation. We define the following **sets**:

- I: set of closing sites (*origins* of the transportation);
- J: set of news sites (*destinations* of the transportation).

The parameters (constants) are the following:

- C_{ij}: unit cost (per rod) for transportation from $i \in I$ to $j \in J$;
- D_i: rods available at origin $i \in I$;
- R_j: rods required at destination $j \in J$;
- F: fixed cost for each truck;
- N: number of trucks;
- L: fixed cost for the rent of an additional truck;
- K: truck capacity.

The decision **variables**:

- x_{ij}: number of rods moved from $i \in I$ to $j \in J$;
- y_{ij}: binary variable that takes value 1 if a truck from $i \in I$ to $j \in J$ is used, 0 otherwise.
- z: binary variable that takes value 1 if the additional truck is used, 0 otherwise.

The model is the following:

$$\min \quad \sum_{i \in I, j \in J} C_{ij} x_{ij} + F \sum_{i \in I, j \in J} y_{ij} + (L - F) z$$

$$\begin{array}{lrcll}
\text{s.t.} & \displaystyle\sum_{i \in I} x_{ij} & \geq & R_j & \forall \ j \in J \\
& \displaystyle\sum_{j \in J} x_{ij} & \leq & D_i & \forall \ i \in I \\
& x_{ij} & \leq & K y_{ij} & \forall \ i \in I, j \in J \\
& \displaystyle\sum_{i \in I, j \in J} y_{ij} & \leq & N + z & \\
& x_{ij} & \in & \mathbb{Z}_+ & \forall \ i \in I, j \in J \\
& y_{ij} & \in & \{0, 1\} & \forall \ i \in I, j \in J \\
& z & \in & \{0, 1\} &
\end{array}$$

The given instance can be obtained by taking $I = \{1, 2, 3\}$, $J = \{A, B, C\}$ and suitable values for the parameters (e.g., $N = 4$).

This structure corresponds to a *transportation problem* and gives the basis for the formulation of problems in several fields. For instance, if sand needs to be moved instead of rods, one only has to change the domain of the x variables from $\mathbb{Z}_+$ to $\mathbb{R}_+$. The model (with the suitable domain for the x variables) might describe the transportation of refrigerators from production centers to shops, or electric energy from power stations to users, etc.

The same model can be modified to describe slightly different problems. For instance, if capacities and/or availabilities are such that a single truck is not sufficient for some origin-destination pairs, one has to decide *how many* trucks should be used for each origin-destination pair. In this case, one needs to:

- introduce new variables $w_{ij} \in \mathbb{Z}_+ \; \forall \, i \in I, j \in J$;
- introduce another family of constraints (counting the number of trucks):

$$x_{ij} \leq K w_{ij} \; \forall \, i \in I, j \in J;$$

- change y with w in the objective function.

Further, let us suppose that there are additional fixed costs for the loading operations at the origins: if some rods had to be moved from a site $i \in I$ to any site in J, then a cost of A has to be paid. The model then would require the following modifications:

- introducing binary variables v_i taking value 1 if loading operations will take place at site $i \in I$, 0 otherwise;
- inserting the following term in the objective function:

$$+ \sum_{i \in I} A \, v_i;$$

- "activating" the binary variables v_i with the constraints

$$\sum_{j \in J} x_{ij} \leq M_i v_i \; \forall \, i \in I$$

  where M is a constant large enough, e.g. $M_i = D_i$. Note that with this definition of the *big-M* parameters, these constraints can replace the constraints on the availability at the origins.

This small example shows how the modeling process of a specific problem can start from a known model (e.g. available in the literature) for a similar problem, where "similar" refers to the structure of the problem rather than the specific practical application. Then one can proceed by making the changes needed to describe exactly the given problem.

2.2 Solution of Exercise 2

A possible formulation is the following:

$$
\begin{array}{llll}
\max & \displaystyle\sum_{j\in J} z_j & & \\
\text{s.t.} & \displaystyle\sum_{i\in I:\sigma_{ij}\geq T} x_i \geq z_j & \forall\, j\in J & \\
 & \displaystyle\sum_{i\in I:\sigma_{ij}\geq T} x_i \leq N + M_j(1-z_j) & \forall\, j\in J & \\
 & x_i \in \{0,1\} & \forall\, i\in I & (1)\\
 & z_j \in \{0,1\} & \forall\, j\in J & (2)
\end{array}
$$

where

- I: set of sites for possible locations;
- J: set of areas;
- σ_{ij}: parameter indicating the signal level of an antenna placed in site $i \in I$ received in area $j \in J$;
- T: parameter indicating the minimum signal level required;
- N: parameter indicating the maximum number of signals above the threshold that a receiver can recognize (in the given instance, $N = 1$);
- x_i: binary variable taking value 1 if an antenna is placed in site $i \in I$, 0 otherwise;
- z_j: binary variable taking value 1 if area $j \in J$ will be covered, 0 otherwise;
- M_j: sufficiently large parameter, e.g., for every area $j \in J$, $M_j = card(\{i \in I : \sigma_{ij} \geq T\})$.

2.3 Solution of Exercise 3 (hints)

A possible model is the following (its generalization is left to the reader):

$$
\begin{array}{ll}
\min & z \\
s.t. & z \geq t_i \qquad \forall i \in A...I \\
& t_A \geq d_A \\
& t_B \geq t_A + d_B - M(1 - y_B) \\
& t_C \geq t_A + d_C - M(1 - y_C) \\
& t_D \geq t_A + d_D \\
& t_E \geq t_B + d_E \\
& t_E \geq t_C + d_E \\
& t_F \geq t_E + d_F - M(1 - y_F) \\
& t_G \geq t_A + d_G - M(1 - y_G) \\
& t_H \geq t_D + d_H \\
& t_H \geq t_E + d_H \\
& t_H \geq t_G + d_H \\
& t_I \geq t_F + d_I + 2y_{CG} \\
& t_I \geq t_G + d_I + 2y_{CG} \\
& y_B + y_C = 1 \\
& y_F + y_G = 1 \\
& y_C + y_G <= 1 + y_{CG} \\
& z, t_i \geq 0 \qquad \forall i \in \{A...I\} \\
& y_\cdot \in \{0, 1\}
\end{array}
$$

where

- t_i variable related to the completion time of operation $i \in \{A, B, C, D, E, F, G, H, I\}$;
- y_i binary variable taking value 1 if operation $i \in \{B, C, F, G\}$ is executed, 0 otherwise;
- y_{CG} binary variable taking value 1 if both C and G are executed, 0 otherwise;
- z variable indicating the completion time of the last operation;
- d_i parameter indicating the duration of operation i;
- M sufficiently large constant.

2.4 Solution of Exercise 4

This problem is similar to a scheduling problem, in which: some *jobs* (persons) have to be processed by different *machines* (newspapers); processing times (reading times) are defined; a specific order for the operations is given; one wants to terminate all operations as soon as possible.

We introduce the following sets:

- I: set of persons;
- K: set of newspapers;

the following parameters:

- D_{ik}: time in minutes needed by person $i \in I$ to read newspaper $k \in K$;
- R_i: time at which person $i \in I$ gets up, in minutes after 8:30 (release time);
- M: a sufficiently large constant, such that M is larger than the optimal completion time, e.g. $M = 60 + \sum_{i \in I, k \in K} D_{ik}$;
- $\sigma[i, l]$: newspaper read by person $i \in I$ in position $l \in \{1, 2...|K|\}$. This parameter defines the reading sequence of each person i. Note: $\sigma[i, l] \in K$ and therefore it can be used as an index for parameters and variables defined on K;

and the following variables:

- h_{ik}: time (in minutes after 8:30) at which person $i \in I$ *starts* to read newspaper $k \in K$;
- y: completion time (in minutes after 8:30);
- x_{ijk}: binary variable taking value 1 if person $i \in I$ reads newspaper $k \in K$ before person $j \in I$, 0 otherwise.

A possible formulation is:

$$
\begin{array}{llll}
\min & y & & (3)\\
\text{s.t.} & y \geq h_{i\,\sigma[i,|K|]} + D_{i\,\sigma[i,|K|]} & \forall\, i \in I & (4)\\
& h_{i\,\sigma[i,l]} \geq h_{i\,\sigma[i,l-1]} + D_{i\,\sigma[i,l-1]} & \forall\, i \in I, l = 2...|K| & (5)\\
& h_{i\,\sigma[i,1]} \geq R_i & \forall\, i \in I & (6)\\
& h_{ik} \geq h_{jk} + D_{jk} - M x_{ijk} & \forall\, k \in K, i \in I, j \in I : i \neq j & (7)\\
& h_{jk} \geq h_{ik} + D_{ik} - M(1 - x_{ijk}) & \forall\, k \in K, i \in I, j \in I : i \neq j & (8)\\
& y \in \mathbb{R}_+ & & (9)\\
& h_{ik} \in \mathbb{R}_+ & \forall\, k \in K, i \in I & (10)\\
& x_{ijk} \in \{0, 1\} & \forall\, k \in K, i \in I, j \in I : i \neq j & (11)
\end{array}
$$

The objective function minimizes the completion time y. Constraint (4), repeated for each person, ensures that everybody actually finishes reading before time y. Constraint (5) guarantees that the reading sequence of each person is satisfied without any overlapping (everybody has to finish reading the previous newspaper before starting the next one). Constraint (6) imposes that nobody starts reading before getting up. Constraints (7) and (8) are disjunctive constraints ensuring that, for each newspaper and for each pair of persons, one of them finishes reading a newspaper before the other starts to read the same newspaper.

2.5 Solution of Exercise 5 (hints)

There are at least two models, whose formulation constitutes a good exercise for the reader.

The first formulations uses the following binary variables:

y_{ik}, taking value 1 if pharmacy i is part of shift k, 0 otherwise;

x_{ijk}, taking value 1 if centroid i uses pharmacy j during shift k.

The second formulation has an exponential number of variables x_J, taking value 1 if the subsets $J \subset P$ (where P is the set of all pharmacies) is chosen as a possible shift, 0 otherwise. Then one has to choose, by means of the x variables, suitable subsets that form a partition of the set P, have balanced cardinality and minimize the total distance covered by the users, shift by shift.

2.6 Solution of Exercise 6

In order to model this problem, we consider a directed graph constructed as follows. Let $G = (N, A)$ be the graph whose nodes correspond to power/distribution stations and whose arcs represent the connections between pairs of stations. For every node $v \in N$, let $b_v = d_v - p_v$ be a parameter representing the difference between the demand that station v has to satisfy and the amount of energy that v can produce. Note that:

- $b_v > 0$ if the station must satisfy a demand that exceeds the production capacity (the station needs to use energy coming from other stations);
- $b_v < 0$ if the production capacity exceeds the demand (the surplus will be conveyed to other stations);
- $b_v = 0$ if the production capacity equals the demand or (as a special case) the station only works as an intermediate node between other pairs of stations ($p_v = d_v = 0$).

In general we can think of b_v as the *demand* of node $v \in N$ (a negative demand represents a surplus that can be sent to the other stations) and classify the nodes as:

- demand nodes (demand $b_i > 0$);

- supply nodes (demand $b_i < 0$);
- transshipment nodes (demand $b_i = 0$).

The decision variables are defined on the arcs of the graph: x_{ij} is the amount of energy that should flow on arc $(i,j) \in A$. The model is as follows:

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{(i,j)\in A} c_{ij}x_{ij} & & & \\
s.t. & \displaystyle\sum_{(i,v)\in A} x_{iv} - \sum_{(v,j)\in A} x_{vj} & = & b_v & \forall v \in N \\
 & x_{ij} & \leq & u_{ij} & \forall (i,j) \in A \\
 & x_{ij} & \in & \mathbb{R}_+ &
\end{array}
$$

The objective function minimizes the total distribution cost, while the constraints define x_{ij} as a *feasible flow* in the graph:

- the first set of constraints, one for each node, is called *node balance constraint* and ensures that the difference between the total flow (of electric energy) entering node v (arcs (i,v)) and the total flow leaving node v (arcs (v,j)) is exactly equal to the demand of the node (which is positive for demand nodes, negative for supply nodes and zero for transshipment nodes);
- the second set of constraints, one for each arc, is the *arc capacity* constraint and bounds the amount of flow on each arc.

We now have a linear programming model for the problem (with no integer variables) and therefore we can use, e.g., the simplex method to solve it.

The above model can be easily extended to other distribution problems on networks: distribution networks for material items (the flow refers to the transportation of the item and the nodes represent production centers, consumption centers and transshipment points), telecommunication networks (the flow represents the bandwidth to be assigned to each connection), transportation networks (the flow represents vehicles in the network), etc. In general, this problem is called *minimum cost network flow problem* and can be used to model and solve many combinatorial optimization problems (that sometimes do not immediately exhibit a network flow structure).

2.7 Solution of Exercise 7

This problem is similar to the previous one, but now there are several flow types, one for each type of energy. Consider a directed graph $G = (N, A)$, where nodes $v \in N$ represent power/distribution stations and arcs $(i, j) \in A$ represent connections between pairs of stations. For every node v and every type of energy k, we define the balance $b_v^k = d_v^k - p_v^k$. Note that, depending on the type of energy, the same node can be a demand node, a supply node and a transsshipment node. Indeed, for every node we have to impose balance constraints for each type of energy. Therefore, we consider decision variables x_{ij}^k indicating the amount of energy of type k flowing on arc (i, j). If K denotes the set of energy types, the model is the following:

$$
\begin{aligned}
\min \quad & \sum_{k \in K} \sum_{(i,j) \in A} c_{ij}^k x_{ij}^k \\
s.t. \quad & \sum_{(i,v) \in A} x_{iv}^k - \sum_{(v,j) \in A} x_{vj}^k = b_v^k \quad \forall\, v \in N,\ \forall\, k \in K \\
& \textstyle\sum_{k \in K} x_{ij}^k \le u_{ij} \quad \forall\, (i, j) \in A \\
& x_{ij}^k \in \mathbb{R}_+ \quad \forall\, (i, j) \in A,\ \forall\, k \in K
\end{aligned}
$$

The objective function minimizes the total distribution cost, obtained by summing the costs over all arcs and all energy types. The first set of constraints imposes balance constraints for all types of energy: for every node v and every energy type k, the sum of the incoming flow (on arcs (i, v)) of type k minus the sum of outgoing flow (on arcs (v, j)) of type k must be equal to the demand of node v (which is positive for demand nodes, negative for supply nodes and zero for transshipment nodes) corresponding to type k. In fact, we have replicated the balance constraint for each energy type. The second set of constraints ensures that arc capacities are not violated: since the capacity is defined for all energy types together (with no distinction), we have one constraint for each arc as in exercise 6, but now the left-hand side takes the sum over *all* energy types.

The model can be generalized by considering, instead of different energy types, different *commodities*. For this reason these problems are called *multicommodity flow problems*: every commodity can be some kind of item on a road network, or a specific traffic demand for an origin-destination pair on a data network, etc.

As in the previous exercise, also in this case having a linear programming model allows one to use standard solution techniques, such as the simplex method. Note that if the arc capacity constraints are ignored, an optimal solution can be obtained by solving $|K|$

minimum cost flow problems separately, one for each commodity. However, the presence of the capacity constraints creates links between the commodities, and therefore one has to deal with them jointly: this makes solving multicommodity flow problems more "difficult" than solving single commodity problems.

2.8 Solution of Exercise 8 (hints)

This problem can be formulated as follows. Define a graph $G = (N, A)$, where the node set N corresponds to the router set and the arc set A corresponds to the set of connection routes. For every traffic demand from node i to node j, define a commodity, thus obtaining the set K of commodities. Therefore for each commodity k we have an origin $o(k)$, a destination $d(k)$ and a traffic demand $r(k)$. Furthermore, for every arc $(i, j) \in A$, we define u_{ij} to be the capacity and c_{ij} the unit cost. If balances for every node $v \in N$ and every commodity $k \in K$ are defined as follows:

$$b_v^k = \begin{cases} -r(k) & \text{if } v = o(k) \\ +r(k) & \text{if } v = d(k) \\ 0 & \text{otherwise} \end{cases}$$

the problem can be modeled as a multicommodity flow problem on graph G.