PROBLEM SHEET 2: MEASURE THEORY
FUNCTIONS THEORY 2024/2025

Exercise 1.
Let μ be a Radon measure on $\mathbb{R}^n$. We define the support of μ as

$$\text{supp } \mu = \mathbb{R}^n \setminus N \qquad \text{where } N = \cup\{A \subseteq \mathbb{R}^n, A \text{ open set}, \mu(A) = 0\}.$$

Prove that $\bar{x} \in \text{supp } \mu$ iff $\int_{\mathbb{R}^n} f(x) d\mu(x) > 0$ for all $f \in C_c(\mathbb{R}^n, [0,1])$ with $f(\bar{x}) > 0$.

Exercise 2. Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1(\mu)$, with $f \geq 0$.

(1) Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if A is a Borel set with $\mu(A) \leq \delta$ then $\int_A f(y) d\mu \leq \varepsilon$. Hint: argue by contradiction. Recall that if $\mu(A) = 0$ then $\int_A f(y) d\mu = 0$.
(2) Show that $\nu(A) := \int_A f(y) d\mu$ is a Radon measure.

Exercise 3. Let $\mathcal{H}^s$ the Hausdorff s measure in $\mathbb{R}^n$. Show that if $s < n$, then $\mathcal{H}^s$ is not a Radon measure.

Exercise 4. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a Lipschitz function (that is there exists $C > 0$, called the Lipschitz constant of f, such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}^n$). Show that for all $s > 0$ and all $A \subseteq \mathbb{R}^n$, there holds

$$\mathcal{H}^s(f(A)) \leq C^s \mathcal{H}^s(A)$$

where C is the Lipschitz constant of f.