Solutions to problems, sheet 5.

Es 1 Let T_u be the distribution associated to u.

Then $\mu = DT_u$ in the sense of distributions

that is $\forall \phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ $\int_{\mathbb{R}^n} \phi(x)\, d\mu(x) = DT_u(\phi) = -T_u(D\phi)$

Reasoning component by component

$\forall \phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ $\int_{\mathbb{R}^n} \phi(x)\, d\mu_i(x) = \partial_{x_i} T_u(\phi) = -T_u(\partial_{x_i}\phi)$

$= -\int_{\mathbb{R}^n} u\, \partial_{x_i}\phi\, dx = -\int_{B(0,R)} u\, \partial_{x_i}\phi\, dx$

if $\operatorname{supp} u \subseteq B(0,R)$

Let $\phi_k \to \phi$ locally in $C^1(\mathbb{R}^n)$ $\Rightarrow$ $\partial_{x_i}\phi_k \to \partial_{x_i}\phi_k$ uniformly in $B(0,R)$

then $\int_{B(0,R)} u \cdot \partial_{x_i}\phi_k \to \int_{B(0,R)} u \cdot \partial_{x_i}\phi\, dx$

$$\Longrightarrow \int_{\mathbb{R}^n} \phi_k \, d\mu_i(x) \longrightarrow \int_{\mathbb{R}^n} \phi \, d\mu_i(x) \quad \forall i = 1 \dots n$$

Let $\phi_k \in C_c^\infty(\mathbb{R}^n, [0,1])$ with $\phi_k = \begin{cases} 1 & \text{on } B(0, R+k) \\ 0 & \text{on } \mathbb{R}^n \setminus B(0, R+k+1) \end{cases}$

$\Longrightarrow$ then $\phi_k \to 1$ locally uniformly in $C^1(\mathbb{R}^n)$

$$\Longrightarrow \int_{\mathbb{R}^n} \phi_k \, d\mu_i(x) \longrightarrow \mu_i(\mathbb{R}^n)$$

$\displaystyle\int_{\mathbb{R}^n} \phi_k \, d\mu_i(x) = -\int_{B(0,R)} \partial_{x_i}\phi_k \cdot u \, dx = 0$ and then $\mu_i(\mathbb{R}^n) = 0$

$\boxed{\text{Ex 2}}$ if $f, g \in C^\infty(\Omega)$ $0 \le f, g \le 1$

$$|\nabla(f + g - fg)| + |\nabla(fg)| = |\nabla f + \nabla g - \nabla f \cdot g - \nabla g \cdot f| + |\nabla f \cdot g + \nabla g \cdot f| =$$

$$= |\nabla f (1-g) + \nabla g (1-f)| + |\nabla f \cdot g + \nabla g \cdot f| \le |\nabla f| (1-g) + |\nabla g| (1-f) +$$

$$+ |\nabla f| \cdot g + |\nabla g| \cdot f = |\nabla f| + |\nabla g| .$$

$$\| |D(f+g-fg)| \|_{L^1} + \| |D(fg)| \|_{L^1} = \| |Df| \|_{L^1} + \| |Dg| \|_{L^1}$$

$f_n \to \chi_E \quad f_n \in C^\infty(\Omega) \quad V(f_n,\Omega) = \|\nabla f_n\|_{L^1} \to P(E,\Omega)$

$g_n \to \chi_F \quad g_n \in C^\infty(\Omega) \quad V(g_n,\Omega) = \|\nabla g_n\|_{L^1} \to P(F,\Omega)$

(Note that $0 \le f_n, g_n \le 1$

$f_n g_n \to \chi_{E\cap F} \qquad \lim_n V(f_n g_n, \Omega) = \|D(f_n g_n)\|_{L^1} \ge P(E\cap F, \Omega)$

$f_n + g_n - f_n g_n \to \chi_{E\cup F} \quad \lim_n V(f_n + g_n - f_n g_n, \Omega) = \|D(f_n + g_n - f_n g_n)\|_{L^1} \ge P(E\cup F, \Omega)$

(up to eventually passing to a subsequence)

Ex 3

1)

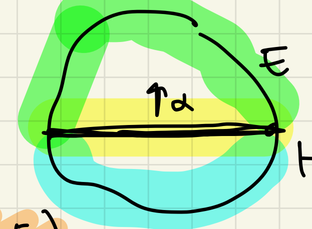
α external normal to H

$$0 = \int_{E\setminus(E\cap H)} \operatorname{div} \alpha = \int_{\partial E\setminus H} \alpha \cdot \nu_E(x)\, dS + \int_{\partial H\cap E} \alpha \cdot (-\alpha)\, dS(x) =$$

$(-\alpha)$ is the exterior normal to $E\setminus(E\cap H)$

$$\le \int_{\partial E\setminus H} dS - \int_{\partial H\cap E} dS = \mathcal{H}^{n-1}(\partial E\setminus H) - \mathcal{H}^{n-1}(\partial H\cap E)$$

$\Rightarrow \mathcal{H}^{n-1}(\partial H \cap E) \leq \mathcal{H}^{n-1}(\partial E \setminus H)$



$$Per(H \cap E) = \mathcal{H}^{n-1}(\partial(H \cap E)) = \mathcal{H}^{n-1}(\partial E \cap H) + \underline{\mathcal{H}^{n-1}(\partial H \cap E)}$$

$$\underset{\text{PREVIOUS COMPUTATION}}{\leq} \mathcal{H}^{n-1}(\partial E \cap H) + \underline{\mathcal{H}^{n-1}(\partial E \setminus H)} = \mathcal{H}^{n-1}(\partial E) = Per(E)$$

2) a convex set $C = \bigcap_{n=1}^{\infty} H_n$ $\quad H_n$ hyperspaces

let $\quad A_1 = H_1 \quad A_2 = H_1 \cap H_2 \quad \dots \quad A_n \subseteq A_{n-1} \quad \bigcap A_n = C$

By 1) $Per(E) \geq Per(E \cap A_1) \geq Per(E \cap A_2) \dots$

$n \mapsto Per(E \cap A_n)$ is a decreasing sequence

$|E \cap A_n| \to |E \cap C|$ in measure and moreover

$P(E \cap A_n) \leq P(E) \Rightarrow P(E) \geq \liminf_n P(E \cap A_n) \underset{\text{by semicontinuity}}{\geq} P(E \cap C)$

$E \cap A_n \supseteq E \cap C$

Ex. 4 $\mathcal{J}(A) \geq 0 \quad \forall A$ of finite perimeter.

$\Rightarrow$ let $c = \inf\limits_{\substack{\chi_A \in BV(\mathbb{R}^N) \\ |A| = m}} \mathcal{J}(A) > 0$

Let E_n be a minimizing sequence so $|E_n| = m \ \forall n$

$$c \leq P(E_n) + \int_{E_n} g(x)\,dx \leq c + \frac{1}{n} \qquad \forall n$$

Note that $\forall R > 0$ $\quad Per(E_n, B_R(0)) \leq Per(E_n) \leq c+1.$

$|E_n \cap B_R(0)| \leq m \quad \longrightarrow \quad \chi_{E_n} \in BV(B_R(0))$ with $\|\chi_{E_n}\|_{BV(B_R(0))} \leq 2(c+1)$

up to a subsequence

$\chi_{E_n} \longrightarrow f^R$ in $L^1(B_R(0))$, since for a.e x $\chi_{E_n}(x) \to f^R(x)$

$\Rightarrow f^R(x) = \chi_{E^R}(x)$. and $\chi_{E^R} \in BV(B_R(0))$

$\chi_{E_{n_1}} \to \chi_{E_1} \quad \to \quad \chi_{E_{n_2}} \to \chi_{E_2}$ in $B(0,2)$ $\qquad E_2 = E_1$ on $B(0,1)$

$\dots \to \quad \chi_{E_{n_n}} \to \chi_E$ locally $\overset{\text{in } L^1}{\curlyvee}$ on compact sets

$\chi_{E_{n_n}} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^N)$ $\qquad \forall R>0 \quad |E \cap B_R(0)| = \lim_n |E_n \cap B_R(0)| \leq m$

$$\lim_{R\to+\infty} |E \cap B_R(0)| = |E \cap \bigcup_R B_R(0)| = |E| \leq m$$

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since $Per(E_n) \leq C+1 \Rightarrow \forall \phi \in C^\infty_c(\mathbb{R}^N)$

$$\int_E \operatorname{div}\phi \, dx = \lim \int_{E_n} \operatorname{div}\phi \, dx \leq \liminf_n Per(E_n) \to \chi_E \in BV(\mathbb{R}^n)$$

$Per(E) \leq \liminf Per(E_n)$

$\downarrow$

since supp ϕ is compact
and $\chi_{E_n} \to \chi_E$ locally in L^1

In order to conclude we only need to prove that

$E_n \to E$ in $L^1(\mathbb{R}^N)$

Indeed if $E_n \to E$ in $L^1(\mathbb{R}^N)$ $\to |E| = m$ since $\|\chi_{E_n}\|_{L^1} \to \|\chi_E\|_{L^1}$ ($\|\chi_{E_n}\|_{L^1} = m$)

Moreover by Fatou lemma (since $g \geq 0$) $\liminf_n \int_{E_n} g(x)\,dx \geq \int_E g(x)\,dx$

$\Longrightarrow$ therefore $\boxed{\mathrm{Per}(E) + \int_E g(x)\,dx = \min_{\substack{|A|=m \\ \chi_A \in BV(\mathbb{R}^N)}} \mathrm{Per}(A) + \int_A g(x)\,dx}$

to prove that $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^N)$ I recall the

Kolmogorov criterium: if $f_n \in L^1(\mathbb{R}^N)$ with

1) $\|f_n\|_{L^1} \leq C$
2) $\lim_{|h| \to 0} \|\tau_h f_n - f_n\|_{L^1} = 0$ uniformly
3) $\forall \varepsilon\ \exists A_\varepsilon$ such that $\|f_n\|_{L^1(\mathbb{R}^N \setminus A_\varepsilon)} \leq \varepsilon$

then up to a subseq $f_n \to f$ in $L^1(\mathbb{R}^N)$

1) is trivially satisfied for χ_{E_m} $\qquad \|\chi_{E_m}\|_{L^1} = m$

For 2, 3 we observe that $\forall R > 0 \quad \exists\, g_R = \min_{|y| \geq R} g(y) \qquad g_R \to +\infty$ as $R \to +\infty$

$$C+1 \geq \int_{\mathbb{R}^N} g(x)\, \chi_{E_m}\, dx \geq \int_{\mathbb{R}^N \setminus B(0,R)} g(x)\, \chi_{E_m}* \geq g_R \|\chi_{E_m}\|_{L^1(\mathbb{R}^N \setminus B(0,R))}$$

so $\quad \|\chi_{E_m}\|_{L^1(\mathbb{R}^N \setminus B(0,R))} \leq \dfrac{C+1}{g_R} \to 0 \qquad$ (and (3) is trivially satisfied)

for $\varepsilon > 0$ take R_ε such that

$$\frac{C+1}{g_{R_\varepsilon}} \leq \varepsilon \quad \to \quad A_\varepsilon = B(0, R_\varepsilon).$$

Moreover

$$\|\tau_a \chi_{E_m} - \chi_{E_m}^*\|_{L^1(\mathbb{R}^N)} \leq \underbrace{\|\tau_a \chi_{E_m} - \chi_{E_m}\|_{L^1(B(0,R))}} + \underbrace{\|\tau_a \chi_{E_m} - \chi_{E_m}\|}_{\leq 2\frac{(C+1)}{g_R}}{}_{L^1(\mathbb{R}^N \setminus B(0,R))}$$

$$\|\tau_h \chi_{E_n} - \chi_{E_n}\|_{L^1(B(0,R))} \le \|\tau_h \chi_{E_n} - \tau_h \chi_E\|_{L^1(B(0,R))} + \|\tau_h \chi_E - \chi_E\|_{L^1(B(0,R))} +$$

$$+ \|\chi_{E_n} - \chi_E\|_{L^1(B(0,R))}$$

Let $\varepsilon > 0$ $\rightarrow$ so take R_ε such that $\dfrac{2(c+1)}{g R_\varepsilon} \le \dfrac{\varepsilon}{4}$

and n_ε such that $\boxed{n \ge n_\varepsilon}$ $\|\tau_h \chi_{E_n} - \tau_h \chi_E\|_{L^1(B(0,R))} \le \dfrac{\varepsilon}{4}$

$\|\chi_{E_n} - \chi_E\|_{L^1(B(0,R))} \le \dfrac{\varepsilon}{4}$

(since $\chi_{E_n} \to \chi_E$ in $L^1(B(0,R))$)

. take $\delta_0 > 0$ such that $\forall |h| \le \delta_0$ $\|\tau_h \chi_E - \chi_E\|_{L^1(B(0,R))} \le \dfrac{\varepsilon}{4}$

(this is true since $\tau_h f \to f$ in $L^1(B(0,R))$ for $f \in L^1(B(0,R))$ if $|h| \le \delta$ as $|h| \to 0$

So for $n \ge n_\varepsilon$ $\|\tau_h \chi_{E_n} - \chi_{E_n}\| \le \varepsilon$ for $|h| \le \delta_0$ (δ_0 depend only on E, R_ε).

moreover since $\tau_h \chi_{E_n} \to \chi_{E_n}$ as $|h| \to 0$ $\forall n$ $\Rightarrow$

let $\delta < \delta_0$ such that $\| \tau_h \chi_{E_n} - \chi_{E_n} \|_{L^1(B(0,R))} \leq \varepsilon$ for

$|h| \leq \delta$ and $n = 1, 2, \dots, n_\varepsilon$.

$\Rightarrow$ we conclude $\forall \varepsilon > 0$ $\exists \delta$ such that $\| \tau_h \chi_{E_n} - \chi_{E_n} \| \leq \varepsilon$
indep of n! for $|h| \leq \delta$.

$\rightarrow$ (2) is satisfied.

So $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^N)$ and E is a minimizer.