Problem sheet 6: Functions of bounded variation

Exercise 1. Let $u \in BV(\mathbb{R}^n)$ with compact support. Let μ_i be the signed valued Radon measure associated to $\frac{\partial}{\partial x_i} T_u$. Show that $\mu_i(\mathbb{R}^n) = 0$ (for every i).

Exercise 2 (Subaddictivity of the perimeter)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be an open set and E, F measurable sets with $\mathrm{Per}(E,\Omega), \mathrm{Per}(F,\Omega) < +\infty$. Show that

$$\mathrm{Per}(E \cup F, \Omega) + \mathrm{Per}(E \cap F, \Omega) \leq \mathrm{Per}(E,\Omega) + \mathrm{Per}(F,\Omega).$$

Hint: Proceed by smooth approximation. Let $f, g \in C^\infty(\Omega)$ with $0 \leq f, g, \leq 1$ and check (pointwise) that

$$|\nabla(f + g - fg)| + |\nabla(fg)| \leq |\nabla f| + |\nabla g|.$$

Exercise 3 (Intersection with convex sets decreases the perimeter)**.** Let $E \subseteq \mathbb{R}^n$ be a bounded set of class C^1 (in particular $\mathcal{H}^{n-1}(\partial E) = \mathrm{Per}E$).

(1) Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $|\alpha| = 1$, and consider the half space $H = \{x \in \mathbb{R}^n \, | x \cdot \alpha < 0\}$. Show that

$$\mathrm{Per}(E \cap H) \leq \mathrm{Per}(E).$$

Hint: Observe that α is the exterior normal to H. If $E \cap H \neq \emptyset$ consider $\int_{E \setminus (E \cap H)} \operatorname{div} \alpha dx = 0$, and then apply the divergence theorem.
The divergence theorem applies to bounded sets Ω which have C^1 (or Lipschitz) boundary up to a possible singular set S with $\mathcal{H}^{n-1}(S) = 0$

(2) Deduce that for $C \subseteq \mathbb{R}^n$ closed and convex there holds that

$$\mathrm{Per}(E \cap C) \leq \mathrm{Per}(E).$$

Hint: recall that C is a the intersection of a countable union of half spaces.

Exercise 4. Let $m > 0$ and $g : \mathbb{R}^n \to \mathbb{R}$ be a positive continuous function with $\lim_{|x| \to +\infty} g(x) = +\infty$. Show that the energy

$$\mathcal{F}(E) = \mathrm{Per}(E) + \int_E g(x) dx$$

admits a minimizer among sets $E \subseteq \mathbb{R}^n$ of finite perimeter such that $|E| = m$.
Hint:use the Kolmogorov criterium for compactness in $L^1(\mathbb{R}^n)$, see the previous sheet).