Ex 1 $\overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,p}} = W_0^{1,p}(\Omega)$ sufficient to prove the ineq. for $C_c^\infty(\Omega)$.

$$|\phi(x', x_n)| = \left|\int_a^{x_n} \frac{\partial}{\partial x_n}\phi(x',t)\,dt\right| \le \int_a^b \left|\frac{\partial}{\partial x_n}\phi(x',t)\right| dt \le$$

$$\le \left[\int_a^b \left|\frac{\partial}{\partial x_n}\phi(x',t)\right|^p\right]^{1/p} |b-a|^{1-\frac{1}{p}}$$

$$|\phi|^p \le |b-a|^{p-1}\int_a^b \left|\frac{\partial}{\partial x_n}\phi\right|^p dx_n$$

$$\int_{\mathbb{R}^{n-1}}\int_a^b |\phi|^p = \|\phi\|_{L^p}^p \le |b-a|^{p-1}\,|b-a| \int_{\mathbb{R}^{n-1}}\int_a^b \left|\frac{\partial}{\partial x_n}\phi\right|^p = |b-a|^p \left\|\frac{\partial}{\partial x_n}\phi\right\|_{L^p}$$

$$\|\phi\|_{L^p} \le |b-a| \left\|\frac{\partial}{\partial x_n}\phi\right\|_{L^p} \le |b-a| \,\|\,|D\phi|\,\|_{L^p}.$$

Ex 2 Observe that if $u_k \to u$ in $L^2(U)$ then

$$\int_U u_k\,dx \to \int_U u\,dx \qquad \left[\text{since } \int_U (u_k - u) \le |U|^{1/2}\|u_k-u\|_{L^2} \to 0\right]$$

So C is weakly closed in $W^{1,2}(U)$

$$E(u) = \int_\Omega |Du|^2 + u \cdot g \overset{\text{Hölder}}{\geq} \| Du \|_2^2 - \|u\|_2 \cdot \|g\|_2 \overset{\text{POINCARÈ}}{\geq}$$

$$\geq \| |Du| \|_2^2 - \|g\|_2 \, C(n,2,\Omega) \cdot \| |Du| \|_2$$

$$\geq - \frac{\|g\|_2^2 \, C(n,2,\Omega)^2}{4}.$$

$$k = \inf E(u) > -\infty$$

Let u_k be a minimizing sequence $\Rightarrow$

$$C+1 \geq c + \frac{1}{k} \geq E(u_k) \geq \| |Du_k| \|_2^2 - \|g\|_2 \, C(n,2,\Omega) \, \| |Du_k| \|_2$$

$$\Longrightarrow \quad \| |Du_k| \|_{L^2} \leq \bar{k} \quad \Longrightarrow \text{Poincaré} \quad \| u_k \|_{W^{1,2}} \leq \tilde{k}$$

$\Rightarrow$ Rellich-Kondrachov $\quad u_{k_i} \longrightarrow u$ in L^2 $\qquad u \in C$

$$\frac{\partial}{\partial x_j} u_{k_i} \longrightarrow \frac{\partial}{\partial x_j} u \quad \text{in } L^2 \quad \Longrightarrow \quad \int u_{k_i} g \to \int u g$$

$$\liminf \int_\Omega |Du_{k_i}|^2 \leq \int_\Omega |Du|^2$$

u is a MINIMIZER.

Ex 3 B compact $\iff$ B sequentially compact

By GLN $B \subseteq B_q = \{u \in L^q(U),\ \|u\|_{L^q} \le C(n,p,q,U)\}$ $\forall q \in [1,p^*]$

Moreover by Rellich Kondrachov $\forall (u_k) \in B$ $\exists\, u_{k_j} \to u$ in $L^q(U)$ $\forall q \in [1,p^*)$. $(p^* = +\infty$ if $p \ge n)$.

If $p > 1 \Longrightarrow u \in W^{1,p}(U)$ and moreover $\begin{cases} u_{k_j} \to u \text{ in } L^p \\ Du_{k_j} \rightharpoonup Du \text{ weakly in } L^p. \end{cases}$

If $p=1$ the limit u in general is NOT in $W^{1,1}(U)$.

$\liminf \|u_k\|_{W^{1,p}(U)} \ge \|u\|_{W^{1,p}(U)} \Longrightarrow u \in B$

by WEAK CONVERGENCE $\liminf \|Du_{k_j}\|_p \ge \|Du\|_p$

So if $1 < p \le n$ B is compact in $L^q(U)$ $\forall q \in [1,p^*)$ (if $p=n$ $p^*=+\infty$).

B is closed in $L^{p^*}(U)$ if $p<n$.

$p>n$ B is compact in $L^q(U)$ $\forall q \in [1,+\infty]$, and also in $C(\overline{U})$, in $C^{0,\alpha}(U)$ $\alpha < 1-\frac{n}{p}$.

Ex 4 Let f_n be a Cauchy sequence in H. Since $V(x) \geq c > 0$ $\inf V$

$\int_{\mathbb{R}^n} |f_n - f_m|^p V(x) \geq c \int_{\mathbb{R}^n} |f_n - f_m|^p dx \Rightarrow f_n$ is Cauchy in $W^{1,p}(\mathbb{R}^n)$

$\Rightarrow f_n \to f$ in $W^{1,p}(\mathbb{R}^n)$

moreover $\int_{\mathbb{R}^n} |f_n|^p V(x) \geq \int_{\mathbb{R}^n} |f|^p V(x) dx$ by Fatou lemma.

$\Rightarrow f \in H$

2 $\|f_k\|_H \leq C \Rightarrow \|f_k\|_{W^{1,p}(B(0,R))} \leq C \quad \forall R \Rightarrow$ by Rellich-Kondrachov $\exists f_{k_i} \to f$ in $L^p(B(0,R))$

fix $R_N \to +\infty$ as $N \to +\infty$

$f_{k,0}$ subseq. converging in $L^p(B(0,R_0))$

$f_{k,1}$ subseq of $f_{k,0}$ converging in $L^p(B(0,R_1))$ ..

$f_{k,N}$ subseq of $f_{k,N-1}$ converging in $L^p(B(0,R_N))$ ..

$f_{k,k}$ is a subseq. converging locally in $L^p(\mathbb{R}^n)$

We check assumptions of Kolmogorov theorem $C \geq \|f_k\|_H$

(1) f_k is bdd in $L^p(\mathbb{R}^n) \Rightarrow C \geq \int_{\mathbb{R}^n} |f_k|^p V(x) \geq C \min V \int_{\mathbb{R}^n} |f_k|^p$.

(3) $\forall \varepsilon > 0$ let $R > 0$ such that $\dfrac{C}{\inf_{|x| \geq R} V(x)} \leq \varepsilon^p$

(Recall $V(x) \to +\infty$ / $|x| \to +\infty$!)

$$C^p \geq \int_{\mathbb{R}^n} |f_k(x)|^p V(x)\, dx \geq \int_{\mathbb{R}^n \setminus B(0,R)} |f_k(x)|^p V(x)\, dx \geq$$

$$\geq \inf_{|x| \geq R} V(x) \, \|f_k\|^p_{L^p(\mathbb{R}^n \setminus B(0,R))} \Rightarrow \left[\begin{array}{l} \|f_k\|_{L^p(\mathbb{R}^n \setminus B(0,R))} \leq \varepsilon \\ \forall k. \end{array} \right.$$

(2) fix $|h| < 1$ fix $\varepsilon > 0$ and let $R > 0$ as above

$$\int_{\mathbb{R}^n} |f_k(x+h) - f_k(x)|^p = \underbrace{\int_{B(0,R+1)} |f_k(x+h) - f_k(x)|^p}_{A} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,R+1)} |f_k(x+h) - f_k(x)|^p}_{B}$$

(B) $$\int_{\mathbb{R}^n\setminus B(0,R+1)} \underbrace{|f_k(x+h)-f_k(x)|^p}_{|f_k(x+h)-f_k(x)|^p \le 2^{p-1}\left[|f_k(x+h)|^p+|f_k(x)|^p\right]} \le 2^p \int_{\mathbb{R}^n\setminus B(0,R)} |f_k(x)|^p\,dx$$

$$\le 2^p \varepsilon^p.\quad \text{(as in point 3 above)}$$

(A) in $B(0,R+1)$ $f_k \to f$ in L^p $\Rightarrow \exists\, k_\varepsilon > 0$ such that

for $k \ge k_\varepsilon$ $\|f_k - f\|_{L^p(B(0,R+1))} \le \varepsilon$

$$\|\tau_h f_k - \tau_h f\|_{L^p} \le \varepsilon$$

$$\|f_k - \tau_h f_k\|_{L^p(B(0,R+1))} \le \|f_k - f\|_{L^p(B(0,R+1))} + \|\tau_h f_k - \tau_h f\|_{L^p\ldots}$$

$$+\|\tau_h f - f\|_{L^p} \le 2\varepsilon + \underbrace{\|\tau_h f - f\|_{L^p(B(0,R+1))}}_{\to 0 \text{ as } |h|\to 0} \qquad k \ge k_\varepsilon$$

$\le \varepsilon$ for $|h| \le \delta_0$

Let $\delta_1 \dots \delta_{k_\varepsilon}$ $\quad \|f_{k_1} - \tau_h f_{k_i}\|_{L^p} \le \varepsilon$ for $|h| \le \delta_{k_i}$ ..

$\to \delta = \min(\delta_0, \delta_1 \dots \delta_{k_\varepsilon})$

$$\|\tau_h f_k - f_k\|^p_{L^p(\mathbb{R}^n)} \le \underbrace{2^p \varepsilon^p}_{B} + 2\varepsilon^p + \varepsilon^p \le \tilde{C}\varepsilon^p$$

$\Rightarrow$ (2) is verified.

So $f_k \to f$ in $L^p(\mathbb{R}^n)$.