Problem sheet 5: Inequalites in Sobolev spaces

Exercise 1 (Poincaré on strips)**.** Let $n > 1$ and consider for $-\infty < a < b < +\infty$,

$$\Omega = \{(x', x_n)\ x' \in \mathbb{R}^{n-1}, a < x_n < b\}.$$

Show that for every $u \in W_0^{1,p}(\Omega)$ for $p \in [1, +\infty)$ there holds

$$\|u\|_p \leq |b-a| \|Du\|_p.$$

Hint: reduce to $u \in C_c^\infty(\Omega)$ and write it as $u(x', x_n) = \int_a^{x_n} \frac{\partial}{\partial x_n} u(x', t)dt$. Then use Hölder.

Exercise 2. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open connected set of class C^1, let $g \in L^2(\Omega)$ with $\int_U g(x)dx = 0$ and consider the energy

$$E(u) = \int_\Omega |\nabla u|^2 + u(x)g(x)dx \qquad u \in W^{1,2}(\Omega).$$

Note that $E(u+k) = E(u)$ for every constant k. Let $C = \{u \in W^{1,2}(\Omega), \int_\Omega u(x)dx = 0\}$. Show that there exists a minimizer of $E(u)$ in C.
Hint: use Hölder inequality and then Poincaré inequality to show that every minimizing sequence is bounded. Then proceed by direct methods.

Exercise 3. Let $U \subseteq \mathbb{R}^n$ be a bounded open set of class C^1. Consider the closed ball

$$B = \{u \in W^{1,p}(U), \|u\|_{W^{1,p}} \leq 1\}.$$

In which functional spaces is this set compact?

Exercise 4 (Compact embedding in $\mathbb{R}^n$)**.** Let $V : \mathbb{R}^n \to (0, +\infty)$ be a continuous function with $\lim_{|x|\to+\infty} V(x) = +\infty$.

For $p \in [1, n)$ we define

$$H = \{u \in W^{1,p}(\mathbb{R}^n), \int_{\mathbb{R}^n} |u(x)|^p V(x)dx < +\infty\}$$

endowed with the norm $\|u\| = \||Du|\|_p + \left(\int |u(x)|^p V(x)dx\right)^{\frac{1}{p}}$.

(1) Show that $(H, \|\cdot\|)$ is a Banach space.
(2) Let f_k be a bounded sequence in H. Show that up to passing to a subsequence $f_k \to f$ in $L^p_{loc}(R^n)$.
Hint: Use Rellich–Kondrachov theorem and diagonalization argument.

We recall the **Kolmogorov compactness theorem** Let $p \in [1, +\infty)$. Assume

- u_k in bounded in $L^p(\mathbb{R}^n)$,
- $\lim_{|h|\to 0} \|\tau_h u_k - u_k\|_p = 0$ uniformly in k
- for all $\varepsilon > 0$ there exists a compact $C_\varepsilon \subseteq \mathbb{R}^n$ such that $\|u_k\|_{L^p(\mathbb{R}^n \setminus C_\varepsilon)} \leq \varepsilon$ for all k.

Then u_k admits a subsequence strongly convergent in $L^p(\mathbb{R}^n)$.

Using this theorem show that the subsequence $f_k \to f$ obtained in item 2 is actually converging in $L^p(\mathbb{R}^n)$. This says that the embedding $H \to L^p(\mathbb{R}^n)$ is compact (also $H \to L^q(\mathbb{R}^n)$ for $q \in [p, p^*)$ is compact).