# Problem sheet 3 (Sobolev spaces)

**Ex 1** (1) Assume that $\exists C>0$ s. that $\forall u \in C^1_c(\mathbb{R})$ $\|u\|_{L^\infty} \le C\|u\|_{[1,p]}$

Let $f \in W^{1,p}(\mathbb{R})$ and $u_n \in C^1_c(\mathbb{R})$ $u_n \to f$ in $W^{1,p}(\mathbb{R})$. Then $u_n$ is a Cauchy sequence in $W^{1,p}(\mathbb{R})$ and $\|u_n - u_m\|_{L^\infty} \le C\|u_n - u_m\|_{[1,p]}$

$\Rightarrow u_n$ is a Cauchy seq. in $L^\infty(\mathbb{R}) \Rightarrow$ it converges to $f$ in $L^\infty(\mathbb{R})$

We conclude by convergence of the norms.

Let $u \in C^1_c(\mathbb{R})$, $x \in \mathbb{R}$ $\quad \left| |u(x)|^{p-1} u(x) \right| = |G(u(x))| = \left| \int_{-\infty}^x \frac{d}{dt} G(u(t))\, dt \right| =$

$$= \left| \int_{-\infty}^x G'(u(t))\, u'(t)\, dt \right| = \left| \int_{-\infty}^x p \underbrace{|u(t)|^{p-1}}_{\in L^{p'}} \underbrace{u'(t)}_{\in L^p} \right| \le \text{Holder} \le p \cdot \|u\|_p^{p-1} \|u'\|_p$$

$$\Rightarrow \quad |u(x)| \le p^{\frac{1}{p}} \|u\|_p^{1-\frac{1}{p}} \|u'\|_p^{\frac{1}{p}} \le \frac{(p\|u'\|_p)}{p} + \frac{\|u\|_p}{p'} = \|u'\|_p + \frac{1}{p'}\|u\|_p$$

$$\le \|u'\|_p + \|u\|_p = \|u\|_{[1,p]}$$

② Since $\phi \in C^1_c(\mathbb{R})$, take $a, b \in \mathbb{R}$ such that $\operatorname{supp}\phi \subset\subset (a,b)$

. Then $\|u_n\|_p^p = \int_{\mathbb{R}} |\phi(x+n)|^p \, dx = \int_a^b |\phi|^p dx \leq \|\phi\|_\infty^p \cdot (b-a)$.

$\|u_n'\|_p^p = \int_{\mathbb{R}} |\phi'(x+n)|^p \, dx = \int_a^b |\phi'|^p dx \leq \|\phi'\|_\infty^p \, (b-a)$.

. Note that $u_n \to 0$ a.e. and so if $u_n \to u$ in $L^q \Rightarrow u = 0$
but $\|u_n\|_q^q = \int_a^b |\phi|^q \, dx > 0$ $\quad q < +\infty$. So $u_n \not\to 0$ in $L^q$.
and on the other hand $\|u_n\|_\infty = \|\phi\|_\infty > 0$. So $u_n \not\to 0$ in $L^\infty$.

## Ex 2 ① Case $p \in (1, +\infty)$

Let $u \in C^1_c(\mathbb{R}^n)$ $\quad \tau_h u - u\,(x) = \int_0^1 \frac{d}{dt} u(x+th)\,dt = \int_0^1 \nabla u\,(x+th) \cdot h$

$|\tau_h u(x) - u(x)|^p = \left|\int_0^1 \nabla u(x+th)\,dt\right|^p |h|^p \leq |h|^p \int_0^1 |\nabla u(x+th)|^p \, dt$ Jensen

$\|\tau_h u - u\|_p \leq |h| \left[\int_{\mathbb{R}^n}\int_0^1 |\nabla u\,(x+th)|^p dt\, dx\right]^{1/p} \leq |h| \, \|\nabla u\|_p$

$u \in W^{1,p}(\mathbb{R}^n)$ $\exists\, u_n \in C^1_c \to u$

$$\|\tau_h u - u\|_p \le \|\tau_h u_n - \tau_h u\|_p + \|\tau_h u_n - u_n\|_p + \|u_n - u\|_p \le$$

$$\le \underbrace{\|\tau_h u_n - \tau_h u\|_p}_{\to 0} + \underbrace{\|u_n - u\|_p}_{\to 0} + \underbrace{|h|\,\|\nabla u_n\|_p}_{\to |h|\,\|\nabla u\|_p} .$$

Moreover if $\bar{\Omega}$ is compact, $\forall\, \omega \subset\subset \Omega$ the same argument above gives $(*)$ $\|\tau_h u - u\|_{L^p(\omega)} \le |h|\,\|\nabla u\|_{L^p(\Omega)}$ for $|h| < \operatorname{dist}(\omega, \partial\Omega)$

(2) $u \in W^{1,\infty}_0(\mathbb{R}^n) \to u \in W^{1,p}_{loc}(\mathbb{R}^n)$. For all $\bar{\Omega}$ compact $(*)$ holds, and as $p \to +\infty$ $\|\tau_h u - u\|_{L^\infty(\omega)} \le |h|\,\|\nabla u\|_{L^\infty(\Omega)} \le |h|\,\|\nabla u\|_{L^\infty(\mathbb{R}^n)}$

so for every compact $\bar{\omega} \subseteq \mathbb{R}^n$ and $\forall |h| > 0 \Rightarrow$

$$\|\tau_h u - u\|_{L^\infty(\omega)} \le |h|\,\|\nabla u\|_{L^\infty(\mathbb{R}^n)}$$

$$\sup_{x \in \omega} |\tau_h u - u| \le |h|\,\|\nabla u\|_{L^\infty(\mathbb{R}^n)} \Rightarrow \text{ by the arbitrariness of } \omega \Rightarrow$$

$$\|\tau_h u - u\|_{L^\infty} \le |h|\,\|\nabla u\|_{L^\infty} .$$

we are using that $\bar{\Omega}$ compact, $f \in L^1 \cap L^\infty(\Omega) \Rightarrow \lim_{p \to +\infty} \|f\|_p = \|f\|_\infty$ since (1) $\|f\|_p \le \|f\|_\infty |\Omega|^{1/p}$ (2) $\forall M > 0$ $A_M = \{x \in \Omega \; |f(x)| \ge M\}$ $\|f\|_p \ge M \cdot |A_M|^{1/p}$.

③ Note that $\forall \phi \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (\tau_h u - u)\, \phi \, dx = \int_{\mathbb{R}^n} (u(x+h) - u(x)) \cdot \phi(x)\, dx = \text{change of variable} =$$

$$= \int_{\mathbb{R}^n} u(x) \cdot [\phi(x-h) - \phi(x)]\, dx$$

Choose $h = te_i$ and divide by $t$ (and send $t \to 0$)

$$\int_{\mathbb{R}^n} u(x)\, \phi_{x_i}(x)\, dx = \lim_{t\to 0} \int_{\mathbb{R}^n} u(x) \left( \frac{\phi(x-te_i) - \phi(x)}{t} \right) dx =$$

$$= \lim_{t \to 0} \int_{\mathbb{R}^n} \left( \frac{u(x+te_i) - u(x)}{t} \right) \cdot \phi(x) \leq \text{Holder} \leq$$

$$\leq \lim_{t \to 0} \left\| \frac{u(\cdot + te_i) - u(\cdot)}{t} \right\|_p \cdot \|\phi\|_{p'} \leq C \|\phi\|_{p'}$$

Consider the functional $T: \phi \longrightarrow \int_{\mathbb{R}^n} u(x)\, \phi_{x_i}(x)$. This is continuous with respect to $L^{p'}$ norm, we may extend to $L^{p'}$ (by Hahn-Banach)

Note $p' \in [1, +\infty)$ !

So by Riesz a linear continuous functional on $L^{p'}$ is associated to a $L^p$ function.

DUAL of $L^{p'}$ is $L^p$ for $p' \in [1, +\infty)$ !

what is $\exists g_i \in L^p(\mathbb{R}^n)$ : $T(\phi) = \int_{\mathbb{R}^n} \phi g_i \, dx \quad \forall \phi \in L^{p'}$

in particular $\forall \phi \in C_c^\infty(\mathbb{R}^n)$

$$\Rightarrow \int_{\mathbb{R}^n} \phi g_i = \int_{\mathbb{R}^n} \phi_{x_i} u \Rightarrow g_i = -u_{x_i} \text{ in weak sense}$$

$$\|g_i\|_p \le C \qquad u \in W^{1,p}(\mathbb{R}^n).$$

Es 3

Let $u \in C_c^2(U)$

$$\|\nabla u\|_2^2 = \int_U \nabla u \cdot \nabla u \, dx := \text{divergence theorem (applied to } u\nabla u) =$$

$$= -\int_U u \cdot \Delta u + \underbrace{\int_{\partial U} u \cdot \nabla u \cdot \vec{n} \, dS}_{=0 \text{ since } u \text{ has compact support}} = -\int_U u \Delta u \le \text{Hölder} \le \|u\|_2 \|\Delta u\|_2$$

Let $u \in H^2(U) \cap H_0^1(U)$ $\quad \exists u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $H^1$.

$\exists v_m \in C^\infty(\bar{U})$ such that $v_m \to u$ in $H^2$.

$$\int_U \nabla u_m \cdot \nabla v_m = -\int_U u_m \cdot \Delta v_m + \underbrace{\int_{\partial U} u_m \cdot \nabla v_m \cdot \vec{n} \, dS}_{=0 \text{ since } u_m \in C_c^\infty(U)} = -\int u_m \cdot \Delta v_m.$$

$$\int_U \nabla u_n \cdot \nabla v_n = \underbrace{\int_U (\nabla u_n - \nabla u)\cdot(\nabla v_n - \nabla u)}_{\to 0} + \int_U \nabla u \cdot \nabla v_n + \int_U \nabla u \, \nabla u_n$$

$$- \int \nabla u \cdot \nabla u \, dx \longrightarrow \int_U \nabla u \cdot \nabla u = \int |\nabla u|^2 dx = \|\nabla u\|_2^2$$

$\nabla u_n \to \nabla u \qquad \nabla v_n \to \nabla u$ in $L^2$ !

$$\int u_n \Delta v_n = \int (u_n - u)\cdot(\Delta v_n - \Delta u) + \int u_n \Delta u + \int u \, \Delta v_n +$$

$$- \int u \Delta u \longrightarrow \int u \cdot \Delta u .$$

$u_n \to u$ in $L^2 \qquad \Delta v_n \to \Delta u$ in $L^2$