Problem sheet 4: Sobolev spaces

Exercise 1 (Sobolev spaces in dimension 1)**.**

(1) Let $p \in [1, +\infty]$. Show that there exists a constant $c > 0$ (independent of p) such that for all $u \in W^{1,p}(\mathbb{R})$ there holds $u \in L^\infty(\mathbb{R})$ with $\|u\|_{L^\infty} \leq c\|u\|_{W^{1,p}}$. So the injection $W^{1,p}(\mathbb{R}) \to L^\infty(\mathbb{R})$ is continuous.
Hint: by density reduce to $u \in C^1_c(\mathbb{R})$. Take $G(u) = |u|^{p-1}u$ and write $G(u(x)) = \int_{-\infty}^x \frac{d}{dt}G(u(t))dt$.

(2) Let $\phi \in C^1_c(\mathbb{R})$ and define $u_n(x) := \phi(x+n)$. Show that u_n is bounded in $W^{1,p}(\mathbb{R})$ for any $p \in [1, +\infty]$ and that is does not admit any converging subsequence in $L^q(\mathbb{R})$ for any possible $q \in [1, +\infty]$.

Exercise 2 (Characterization of Sobolev spaces)**.** Let $u \in L^p(\mathbb{R}^n)$. Define $\tau_h u(x) := u(x+h)$.

(1) Show that if $u \in W^{1,p}(\mathbb{R}^N)$ for $p \in [1, +\infty)$ then

$$\|\tau_h u - u\|_p \leq |h|\|\nabla u\|_p.$$

Deduce that if $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ for $p \in [1, +\infty)$ then for all open bounded set Ω and all $\omega \subset\subset \Omega$, there holds

$$\|\tau_h u - u\|_{L^p(\omega)} \leq |h|\|\nabla u\|_{L^p(\Omega)} \qquad \text{for } |h| \leq \text{dist}(\omega, \partial\Omega).$$

Hint: reduce to smooth functions and write $\tau_h u(x) - u(x) = \int_0^1 \frac{d}{dt}u(x+th)dt$. Recall Jensen inequality: for ϕ convex $\phi(\frac{1}{b-a}\int_a^b f(t)dt) \leq \frac{1}{b-a}\int_a^b \phi(f(t))dt$.

(2) Show that if $u \in W^{1,\infty}(\mathbb{R}^N)$ then

$$\|\tau_h u - u\|_\infty \leq |h|\|\nabla u\|_\infty.$$

So $u \in W^{1,\infty}(\mathbb{R}^N)$ has a representative which is a Lipschitz continuous function.
Hint: Observe that if $u \in W^{1,\infty}(\mathbb{R}^N)$ then $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ for all $p \leq +\infty$. Now use the fact that $\lim_{p\to+\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}$ if $f \in L^q(\Omega)$ for all $q \leq +\infty$ for Ω bounded open set.

(3) Let $p \in (1, +\infty]$. Assume there exists $C > 0$ such that

$$\|\tau_h u - u\|_p \leq C|h|$$

Show that $u \in W^{1,p}(\mathbb{R}^N)$ and $C \geq \|\nabla u\|_p$.
Hint: Consider $\int_{\mathbb{R}^N} \frac{u(x+te_i)-u(x)}{t}\phi(x)dx$ for some $\phi \in C^\infty_c(\mathbb{R}^n)$. Apply Hölder, and show that for every i, $\int_{\mathbb{R}^n} u\phi_{x_i}dx \leq C\|\phi\|_{p'}$.

Exercise 3. Let U be an open bounded set with C^1 boundary in $\mathbb{R}^n$. Show that for all $u \in W^{1,2}_0(U) \cap W^{2,2}(U)$ there holds

$$\|\nabla u\|_2^2 \leq \|u\|_2\|\Delta u\|_2$$

where $\Delta u = \text{div}\nabla u$ (in weak sense).
Hint: Recall the density result and the definition of $W^{1,p}_0$. Integrate by parts.