

PROBLEM SHEET 4: SOBOLEV SPACES

**Exercise 1** (Sobolev spaces in dimension 1).

- (1) Let  $p \in [1, +\infty]$ . Show that there exists a constant  $c > 0$  (independent of  $p$ ) such that for all  $u \in W^{1,p}(\mathbb{R})$  there holds  $u \in L^\infty(\mathbb{R})$  with  $\|u\|_{L^\infty} \leq c\|u\|_{W^{1,p}}$ . So the injection  $W^{1,p}(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$  is continuous.  
 Hint: by density reduce to  $u \in C_c^1(\mathbb{R})$ . Take  $G(u) = |u|^{p-1}u$  and write  $G(u(x)) = \int_{-\infty}^x \frac{d}{dt} G(u(t)) dt$ .
- (2) Let  $\phi \in C_c^1(\mathbb{R})$  and define  $u_n(x) := \phi(x+n)$ . Show that  $u_n$  is bounded in  $W^{1,p}(\mathbb{R})$  for any  $p \in [1, +\infty]$  and that it does not admit any converging subsequence in  $L^q(\mathbb{R})$  for any possible  $q \in [1, +\infty]$ .

**Exercise 2** (Characterization of Sobolev spaces). Let  $p \in (1, +\infty]$  and  $u \in L^p(\mathbb{R}^n)$ . Define  $\tau_h u(x) := u(x+h)$ .

- (1) Show that if  $u \in W^{1,p}(\mathbb{R}^N)$  for  $p \in (1, +\infty)$  then

$$\|\tau_h u - u\|_p \leq |h| \|\nabla u\|_p.$$

Deduce that if  $u \in W_{loc}^{1,p}(\mathbb{R}^N)$  for  $p \in (1, +\infty)$  then for all open bounded set  $\Omega$  and all  $\omega \subset\subset \Omega$ , there holds

$$\|\tau_h u - u\|_{L^p(\omega)} \leq |h| \|\nabla u\|_{L^p(\Omega)} \quad \text{for } |h| \leq \text{dist}(\omega, \partial\Omega).$$

Hint: reduce to smooth functions and write  $\tau_h u(x) - u(x) = \int_0^1 \frac{d}{dt} u(x+th) dt$ . Recall Jensen inequality: for  $\phi$  convex  $\phi(\frac{1}{b-a} \int_a^b f(t) dt) \leq \frac{1}{b-a} \int_a^b \phi(f(t)) dt$ .

- (2) Show that if  $u \in W^{1,\infty}(\mathbb{R}^N)$  then

$$\|\tau_h u - u\|_\infty \leq |h| \|\nabla u\|_\infty.$$

So  $u \in W^{1,\infty}(\mathbb{R}^N)$  has a representative which is a Lipschitz continuous function.

Hint: Observe that if  $u \in W^{1,\infty}(\mathbb{R}^N)$  then  $u \in W_{loc}^{1,p}(\mathbb{R}^N)$  for all  $p \leq +\infty$ . Now use the fact that  $\lim_{p \rightarrow +\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}$  if  $f \in L^q(\Omega)$  for all  $q \leq +\infty$  for  $\Omega$  bounded open set.

- (3) Let  $p \in (1, +\infty]$ . Assume there exists  $C > 0$  such that

$$\|\tau_h u - u\|_p \leq C|h|$$

Show that  $u \in W^{1,p}(\mathbb{R}^N)$  and  $C \geq \|\nabla u\|_p$ .

Hint: Consider  $\int_{\mathbb{R}^N} \frac{u(x+te_i) - u(x)}{t} \phi(x) dx$  for some  $\phi \in C_c^\infty(\mathbb{R}^n)$ . Apply Hölder, and show that for every  $i$ ,  $\int_{\mathbb{R}^n} u \phi_{x_i} dx \leq C \|\phi\|_{p'}$ .

**Exercise 3.** Let  $U$  be an open bounded set with  $C^1$  boundary in  $\mathbb{R}^n$ . Show that for all  $u \in W_0^{1,2}(U) \cap W^{2,2}(U)$  there holds

$$\|\nabla u\|_2^2 \leq \|u\|_2 \|\Delta u\|_2$$

where  $\Delta u = \text{div} \nabla u$  (in weak sense).

Hint: Recall the density result and the definition of  $W_0^{1,p}$ . Integrate by parts.