Sketch of solutions, sheet 3

Ex 1

$$\partial_i T_{\chi_E}(\varphi) = -T_{\chi_E}(\partial_i \varphi) = -\int_E \partial_i \varphi \, dx =$$

$$= \text{divergence theorem} = -\int_{\partial E} \varphi(x) \underbrace{\nu_i(x)}_{} \, d\mathcal{H}^{n-1}(x)$$

i-th component of the exterior normal at x to E.

$$\nabla T_{\chi_E}(\varphi) = -\int_{\partial E} \varphi(x) \nu_E(x) \, d\mathcal{H}^{n-1}(x)$$

where ν_E is the exterior normal at x to E.

Ex 2 Let $\phi \in C^\infty_c$. Assume supp $\phi \subset\subset [-a,a]$, and let $\{x_1 \dots x_k\}$ be the jump discontinuities of f contained in $[-a,a]$

$$-\int \phi f' = \int \phi' f \, dx = \int_{-a}^{x_1} \phi' f + \int_{x_1}^{x_2} \phi' f + \dots + \int_{x_{k-1}}^{x_k} \phi' f + \int_{x_k}^{a} \phi' f = \text{integration by}$$

$$\text{parts} = -\int_{-a}^{x_1} \phi \frac{df}{dx} + \phi(x_1) f^-(x_1) - \int_{x_1}^{x_2} \phi \frac{df}{dx} + \phi(x_2) f^-(x_2) - \phi(x_1) f^+(x_1) \dots +$$

$$-\int_{x_k}^{a} \phi \frac{df}{dx} - \phi(x_k) f^+(x_k) = -\int_{-a}^{a} \phi \frac{df}{dx} + \sum_{i=1}^{k} \phi(x_i) \cdot (f(x_i)^- - f(x_i)^+)$$

Ex 3 If $T = c$ then obviously $T' = 0$.

Assume now $T(\phi') = 0 \ \forall \ \phi \in C^\infty_c(I)$. Fix $\psi_0 \in C^\infty_c(I)$ $\int_I \psi_0 \, dx = 1$ $T\psi_0 = c \in \mathbb{R}$

consider $h(x) = \phi(x) - \psi_0(x) \int_I \phi(y) dy$. then $h \in C^\infty_c(I)$ and $\int_I h = 0$

so $w(x) = \int_0^x \phi(y) - \psi_0(y) \left(\int_I \phi(t) dt \right) dy \in C^\infty_c(I)$ and $w'(x) = h(x)$.

By ass. $0 = T(w') = T(\phi(x) - [\int_I \phi(y) dy] \psi_0(x)) = T\phi - \left(\int \phi(y) \, dy \right) T\psi_0 = \Delta$

$$T(\phi) = T(\psi_0) \int_I \phi(y) dy = c \int_I \phi(y) dy \Rightarrow T = c.$$

Ex 4 Fix $K \subset\subset U$ compact

Fix $\phi \in C_c^\infty(U)$ $\phi \equiv 1$ on K $0 \leq \phi \leq 1$

Then $\forall \psi \in C_c^\infty(U)$ with $\operatorname{supp} \psi \subseteq K$ we have that

$$-\|\psi\|_\infty \phi \leq \psi \leq \|\psi\|_\infty \phi \xRightarrow{\text{by POSITIVITY of } T} -\|\psi\|_\infty T(\varphi) \leq T(\psi) \leq \|\psi\|_\infty T(\varphi)$$

$$\Longrightarrow \quad |T(\psi)| \leq \|\psi\|_\infty T(\varphi)$$

So T has order 0.

Ex 5 ϕ_ε with $\operatorname{supp} \phi_\varepsilon \subseteq [0,1]$ $\phi_\varepsilon = 1$ for $x \in (\varepsilon^2, \varepsilon)$

$$T(\varphi_\varepsilon) = \lim_{\delta \to 0} \int_\delta^1 \frac{\phi_\varepsilon(x)}{x}\, dx \geq \lim_{\delta \to 0} \int_{\varepsilon^2}^{\varepsilon} \frac{1}{x}\, dx = \log \varepsilon - \log \varepsilon^2 =$$

$$= \log \frac{1}{\varepsilon} \to +\infty$$

$$\log \frac{1}{\varepsilon} = |T(\varphi_\varepsilon)| \leq C\, |\varphi_\varepsilon|_\infty = C \qquad (\text{Absurd!})$$