Problem sheet 3: Distribution theory
Functions theory 2024/2025

Exercise 1. Let $N > 1$ and $E \subseteq \mathbb{R}^N$ be a bounded open set with boundary of class C^1 and consider the distribution
$$T_{\chi_E}(\phi) := \int_E \phi(x)dx.$$
Compute the distributional gradient of T_{χ_E}.

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$, such that $f \in C^1(\mathbb{R}\setminus\{x_1, \dots, x_m\})$, f has jump discontinuities in $\{x_1, \dots, x_m\}$, and the classical derivative (defined in $\mathbb{R} \setminus \{x_1, \dots, x_m\}$) satisfies $\frac{df}{dx} \in L^1_{\text{loc}}(\mathbb{R})$. Show that the derivative of f in the sense of distributions coincides with
$$f' = \frac{df}{dx} + \sum_{i=1}^m (f(x_i)^+ - f(x_i)^-)\delta_{x_i}$$
where $f(x_i)^+ - f(x_i)^- = \lim_{x\to x_i^+} f(x) - \lim_{x\to x_i^-} f(x)$ and δ_{x_i} is the distribution which associates to ϕ the value $\phi(x_i)$.

Exercise 3. Let $T \in \mathcal{D}'(I)$ for some interval $I \subseteq \mathbb{R}$. Show that $T' = 0$ in the sense of distribution (this means $T'\phi = -\int T\phi' dx = 0$) if and only if there exists a constant c and $T = c$ (this means $T\phi = c\int \phi dx$).
Hint: argue as in the fundamental lemma of the calculus of variations.

Exercise 4.
Let $T \in \mathcal{D}'(U)$ be a positive distribution, that is $T(\phi) \geq 0$ for every test $\phi \geq 0$. Prove that T has order 0.
Hint: recall the property of positive linear functionals on $C_c(U)$

Exercise 5. Show that the distribution principal value of $\frac{1}{x}$ in $\mathbb{R}$ is of order 1.
Hint: it is trivial to check that it is of order 0 or 1. For $\varepsilon > 0$, take a test function ϕ_ε, with $\text{supp}\phi_\varepsilon = [0, 1]$, $0 \leq \phi_\varepsilon \leq 1$ and $\phi_\varepsilon(x) = 1$ for $x \in (\varepsilon^2, \varepsilon)$ ...