Sketch of solutions, sheet 2 (Measure theory)

Es 1

Observe that if $x \in N$, every A open containing x satisfies $\mu(A) > 0$

Let f such that $f(\bar{x}) > 0 \to \exists A$ open containing $\bar{x}$ such that $f(x) \geq \frac{f(\bar{x})}{2} > 0 \quad \forall x \in A$

If $x \notin \operatorname{supp} \mu$, $\exists A$ open s.t. $\mu(A) = 0$ and $x \in A$. Let $r > 0$ $\overline{B(x,r)} \subseteq A$. Construct f s.t. $f = 1$ on $\overline{B(x,r)}$, $f = 0$ in $\mathbb{R}^n \setminus A$, $f(x) \in [0,1] \quad \forall x$

then $\int_{\mathbb{R}^n} f(x)\,d\mu = \int_A f(x)\,d\mu = 0$

Es 2

① By contradiction $\exists \varepsilon > 0$ $\forall n > 0$ $\exists E_n$, $\mu(E_n) \leq \frac{1}{2^n}$.

$\int_{E_n} f(y)\,d\mu > \varepsilon$ Let $F_k = \bigcup_{n \geq k} E_n$ $F = \bigcap_k F_k$

$$\mu(F_k) \leq \sum_{n \geq k} \mu(E_n) = \sum_{n \geq k} \frac{1}{2^n} = 2^{1-k}$$

$F_{k+1} \subseteq F_k \dots \Rightarrow \mu(F) = 0$ but $\int_{F_k} f(y)\,d\mu \geq \varepsilon \ \forall k$

$\Rightarrow \int_F f(y)\,d\mu \geq \varepsilon$ absurd (since $\int_F f(y)\,d\mu = 0$)

2) σ-additivity A_i disjoint Borel sets

$$\phi_k(x) = \sum_1^k \chi_{A_i}(x) = \chi_{\bigcup_i A_i} \Rightarrow \phi(x) = \chi_{\bigcup_i A_i} = \lim_k \phi_k(x)$$

$0 \leq f(x)\phi_k(x) \leq f(x)\phi_{k+1}(x)$ → monotone convergence $\Rightarrow$

$\bigcup_i \nu(A_i)$

$$\parallel \lim_k \sum_{i=1}^k \int_{A_i} f(x)\,d\mu = \lim_k \int f(x)\phi_k(x)\,dx = \int f(x)\phi(x) = \int_{\bigcup_i A_i} f(x)\,dx$$

$= \nu(\bigcup_i A_i)$

$\Longrightarrow \quad \sum_i \nu(A_i) = \nu(\bigcup_i A_i)$

Since f is in $L^1(\mu)$, ν is finite $\Rightarrow$ then also locally finite

[by (1) $\nu \ll \mu$.]

Es 3 let K be a compact set in $\mathbb{R}^n$ with $|K|>0 \Longrightarrow$

$\mathcal{H}^n(K) > 0 \Longrightarrow \mathcal{H}^s(K) = +\infty \quad \forall \quad s < n$. $\square$

Es 4 let $\delta > 0$ and let $C_i \subseteq \mathbb{R}^n$ diam $C_i \leq \delta$ $A \subseteq \bigcup_i C_i$

$\Rightarrow$ diam $f(C_i) \leq C\delta$ and $f(A) \subseteq \bigcup_i f(C_i)$

$$\Rightarrow \mathcal{H}^s_{C\delta}(f(A)) \leq \sum_{i=1}^{\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} f(C_i))^s \leq C^s \sum_1^{\infty} \frac{\alpha(s)}{2^s} (\operatorname{diam} C_i)^s$$

$\rightarrow$ taking the infimum among all covers $\Rightarrow$

$\mathcal{H}^s_{C\delta} f(A) \leq C^s \cdot \mathcal{H}^s_\delta(A) \Rightarrow$ conclude sending $\delta \to 0$.