PROBLEM SHEET 1: PRELIMINARIES
FUNCTIONS THEORY 2024/2025

Exercise 1. Let $U \subseteq \mathbb{R}^n$ be an open bounded set. Let $C^{0,\alpha}(U)$ for $\alpha \in (0,1]$ be the space of Hölder continuous functions of exponent α, so $u \in C(\bar{U})$ and there exists $C > 0$ such that $|u(x) - u(y)| \leq C|x-y|^\alpha$ for all $x, y \in U$. We define the norm $\|u\|_\alpha = \|u\|_\infty + \sup_{x \neq y \in U} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.

(1) Show that $(C^{0,\alpha}(U), \|u\|_\alpha)$ is a Banach space.
Hint: show that $C(\bar{U}), \|\cdot\|_\infty)$ is a closed subspace of $(L^\infty(U), \|\cdot\|_\infty)$.

(2) Show that if $u_n \in C^{0,\alpha}(U)$ is a sequence with $\|u_n\|_\alpha \leq C$, then up to a subsequence, $u_n \to u$ in $C^{0,\beta}(U)$, where $u \in C^{0,\alpha}(U)$ (that is the immersion $C^{0,\alpha}(U) \to C^{0,\beta}(U)$ is compact for every $\beta < \alpha$).
Hint: use the Ascoli-Arzelà compactness theorem.

Exercise 2 (Hardy's inequality)**.** Let $u \in C^1(\overline{B(0,r)})$ where $B(0,r) \subseteq \mathbb{R}^n$ the ball of center 0 and radius r. Assume that the dimension of the space is $n \geq 3$.

(1) Show that

$$\frac{1}{r}\int_{\partial B(0,r)} u^2 dS = \frac{1}{r^2}\int_{B(0,r)} (nu^2 + 2u\nabla u \cdot x)dx \leq \int_{B(0,r)} \left(\frac{n+1}{r^2}u^2 + |\nabla u|^2\right) dx.$$

Hint: Apply divergence theorem to xu^2 and then Young inequality (that is $2ab \leq a^2/c + cb^2$, for every $c > 0$).

(2) Observing that $\operatorname{div}\frac{x}{|x|^2} = \frac{n-2}{|x|^2}$ and using divergence theorem, show that for $\varepsilon \in (0,r)$

$$\int_{B(0,r)\setminus B(0,\varepsilon)} (n-2)\frac{u^2}{|x|^2}dx = -\int_{B(0,r)\setminus B(0,\varepsilon)} 2u\nabla u \cdot \frac{x}{|x|^2}dx + \frac{1}{r}\int_{\partial B(0,r)} u^2 dS - \frac{1}{\varepsilon}\int_{\partial B(0,\varepsilon)} u^2 dS.$$

Conclude that, for $\delta \in (0, n-2)$

$$\int_{B(0,r)\setminus B(0,\varepsilon)} (n-2-\delta)\frac{u^2}{|x|^2}dx \leq \frac{1}{\delta}\int_{B(0,r)} |\nabla u|^2 dx + \frac{1}{r}\int_{\partial B(0,r)} u^2 dS - \frac{1}{\varepsilon}\int_{\partial B(0,\varepsilon)} u^2 dS.$$

Hint: recall Young inequality.

(3) Using 1, and 2, prove that $\frac{u(x)}{|x|} \in L^2(B(0,r))$ and there exists $C = C(n)$ such that

$$\int_{B(0,r)} \frac{u^2(x)}{|x|^2}dx \leq C\int_{B(0,r)} \left(\frac{u^2(x)}{r^2} + |\nabla u|^2\right) dx.$$

(4) Show that if $n = 1, 2$ and $u(0) \neq 0$ then $\frac{u(x)}{|x|} \notin L^2(B(0,1))$.

Finally show that if $u \in C^1(\mathbb{R}^n)$, $n \geq 3$ with $u, |\nabla u| \in L^2(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2}dx \leq \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

Hint: use 2, with $\delta = \frac{n-2}{2}$.