FUNCTIONS THEORY, A.A. 2024/2025.

DETAILED PROGRAM

Preliminaries and measure theory. References: [F], Appendix [E]
Spaces of Hölder continuous functions. Ascoli Arzelà compactness theorem.
Regularization by convolution.
The fundamental lemma of calculus of variations (Du Bois-Reymond lemma) and corollary in dimension 1.
Partition of units subordinated to an open covering.
Gauss-Green formula and divergence theorem, integration on sphere formula, computation of the volume of the n dimensional ball.
Open set of class C^k. Signed distance function from a set. Sets of class C^k, $k \geq 2$ and extension of the normal.

Geometric measure theory. References: [EG], [F].
Definition of Radon measures. Radon measures are regular measures (no proof).
Outer measure and Caratheodory criterium (no proof).
Definition of signed measure, vector valued measure, total variation measure.
Absolutely continuous measures with respect to Lebesgue and mutually singular measures.
Radon Nikodym decomposition. Differentiation of measures. Lebesgue differentiation theorem and measure theoretic boundary of a set (no proofs)
Vitali and Besikovitch covering theorem (no proof).
Hausdorff measures: definition, properties and Hausdorff dimension of a set. Example of the Cantor set. Isodiametric inequality and proof of the fact that $\mathcal{H}^n$ coincides with Lebesgue measure.
Countable k-rectifiable sets and k-rectifiable sets. Purely k-unrectifiable sets.
Positive linear functionals on $C_c(U)$ and the Riesz Markov theorem (no proof). Generalization.
Linear continuous functionals in $(C_0(U), \|\cdot\|_\infty)$ and space of Radon measures. Weak convergence in the space of Radon measures.

Distribution theory. References: [F]
The space of test functions $C_c^\infty(U)$ and the notion of convergence on it.
Definition of a distribution and notion of convergence in the space of distributions. Relation between convergence in the space of distribution and weak convergence in L^p.
Order of a distribution. Examples of distributions. Principal value distribution.
Support of a distribution. Distributions with compact support have finite order.
Derivatives in the sense of distributions. Relation between classical derivatives, derivatives a.e., weak derivatives and derivatives in the sense of distributions. Monotone functions in $\mathbb{R}$ have distributional derivative which is positive (as a distribution), so a Radon measure.
Differential operators and fundamental solutions. Fundamental solution of the laplacian operator.
Convolution between a distribution and a test function. Density of test functions in the space of distributions.
Weyl lemma (harmonic distributions are smooth functions).

Sobolev spaces. References: [E], [EG].
Definition of Sobolev spaces $W^{k,p}(U)$ and of the norm $\|\cdot\|_{k,p}$. Sobolev space are Banach spaces (case $k = 1$). The space $W_0^{k,p}(U)$. Examples.
Sobolev spaces in 1 dimension: characterization of the functions in $W^{1,p}(I)$, I open bounded interval. Compact embedding of $W^{1,p}(I)$ in $L^\infty(I)$ for $p > 1$, counterexample with $p = 1$.
Density of smooth functions in $W^{1,p}(\mathbb{R}^n)$. Meyers Serrin theorem. Density of $C^\infty(\bar{U})$ for U bounded open set of class C^1 (no proof.).
Definition of extension. Extension theorem (no proof). Counterexample to extension (with a domain with Hölder boundary). Trace operator and characterization of $W_0^{1,p}(U)$ (no proof).
Definition of Sobolev conjugate. Gagliardo-Nirenberg -Sobolev inequality. Limit case $p = n$ (sketch). Local version of (GNS) inequality for $W_0^{1,p}(U)$ and for $W^{1,p}(U)$.

Morrey inequality. Local case and compact embedding in Hölder spaces.
Functions in $W^{1,\infty}_{loc}$ are locally Lipschitz and viceversa (sketch).
Differentiability a.e. of functions in $W^{1,p}_{loc}$ for $p \in (n, +\infty]$ (generalized Rademacher theorem).
Continuous embeddings of $W^{2,p}$ spaces.
Rellich Kondrachov theorem. Counterexample to the limit case $q = p^*$.
Corollaries of Rellich Kondrachov: compactness criterium in $W^{1,p}(U)$ (statement for $W^{1,p}(U)$, $p = 1$ and $p > 1$).
Poincaré inequality. Embedding of $W^{1,n}(\mathbb{R}^n)$ in BMO (sketch).
Applications to calculus of variations problems. The Dirichlet integral, harmonic extension.

Functions of bounded variation. References: [AFP], [EG].
Definition of the space BV and of the BV norm. Lower semicontinuity of the total variation with respect to L^1 convergence. The space BV is Banach. $W^{1,1}$ is continuously embedded in BV.
Sets of finite perimeter (or Caccioppoli sets): definition. Case of C^k sets of $k > 1$.
BV functions in dimension 1. Definition of pointwise variations. Functions with bounded pointwise variations are difference of monotone functions. Essential pointwise variations and coincidence with the total variation (no proof). Characterization of BV functions in 1 dimension (no proof): absolutely continuous part, jump part and cantorian part. Example of the Cantor function.
Strict convergence and weak$*$ convergence in BV. Approximation in strict sense of smooth functions (no proof).
Compactness theorem in BV (Helly's theorem). Local compactness for sequence of sets with finite perimeter. Applications: the Plateau problem.
Gagliardo-Nirenberg-Sobolev inequality and Poincaré-Wirtinger inequality for BV functions. Isoperimetric and local isoperimetric inequality for sets of finite perimeter.

References.

[AFP] Ambrosio, L., Fusco, N., and Pallara, D., Functions of bounded variations and free discontinuity problems, Oxford Mathematical Monographs, 2001.
[E] Evans, L.C., Partial Differential Equations, American Mathematical Soc., 2010.
[EG] Evans L.C.,and Gariepy, F.C., Measure theory and fine properties of functions, Boca Raton, CRC Press, 1992.
[F] Folland, G.B., Real Analysis. New York: Wiley Interscience, 1999.