Soluzioni integrali generalizzati

1) $$\int_0^{+\infty} \frac{1}{e^{2x}+1}\,dx = \lim_{M\to+\infty} \int_0^M \frac{1}{e^x+1}\,dx$$

Calcolo $\int_0^M \frac{1}{e^x+1}\,dx$ cambio variabile $y=e^x$ $\quad x=\lg y$

$dx=\frac{1}{y}\,dy$

$x=0 \to y=e^0=1$

$x=M \to y=e^M$

$$\int_1^{e^M} \frac{1}{(y+1)}\cdot\frac{1}{y}\,dy = \int_1^{e^M} \frac{1}{(y+1)\cdot y}\,dy$$

$$\frac{1}{(y+1)\cdot y} = \frac{A}{y+1} + \frac{B}{y} = \frac{Ay+By+B}{(y+1)y}$$

$$\begin{cases} A+B=0 \\ B=1 \end{cases} \qquad \begin{matrix} A=-1 \\ B=1 \end{matrix}$$

$$\int \frac{1}{(y+1)\cdot y}\,dy = \int \frac{-1}{y+1}\,dy + \int \frac{1}{y}\,dy = -\lg|y+1| + \lg|y| + c =$$

$$= \lg\frac{|y|}{|y+1|} + c$$

$$\int_1^{e^M} \frac{1}{(y+1)y}\,dy = \left[\lg\left|\frac{y}{y+1}\right|\right]_1^{e^M} = \lg\left(\frac{e^M}{e^M+1}\right) - \lg\frac{1}{2}$$

$$\lim_{M\to+\infty} \int_0^M \frac{1}{e^x+1}\,dx = \lim_{M\to+\infty} \lg\left(\frac{e^M}{e^M+1}\right) - \lg\frac{1}{2} = -\lg\frac{1}{2} = \lg 2$$

$$\frac{e^M}{e^M+1} \to 1$$

$$\frac{e^M}{e^M+1} = \frac{\cancel{e^M}}{\cancel{e^M}\left(1+\frac{1}{e^M}\right)} = \frac{1}{\left(1+\frac{1}{e^M}\right)}$$

2) $$\int_1^{+\infty} \frac{5}{x\left(14+9\lg x+(\lg x)^2\right)}\,dx = \lim_{M\to+\infty} \int_1^M \frac{5}{x\left(14+9\lg x+(\lg x)^2\right)}\,dx$$

$$y = \lg x \qquad dy = \frac{1}{x}\,dx$$

$x=1 \to y=\lg 1=0$

$x=M \to y=\lg M$

$$\int_0^{\lg M} \frac{5}{14+9y+y^2}\,dy$$

$$14+9y+y^2=0 \qquad y_{1,2} = \frac{-9\pm\sqrt{81-56}}{2} = \frac{-9\pm 5}{2} \begin{cases} -7 \\ -2 \end{cases}$$

$$\frac{5}{y^2+9y+14} = \frac{A}{y+2} + \frac{B}{y+7} = \frac{Ay+7A+By+2B}{(y+2)(y+7)} \qquad \begin{cases} A+B=0 \\ 7A+2B=5 \end{cases}$$

$$A=1$$
$$B=-1$$

$$\int \frac{5}{y^2+9y+14}\,dy = \int \frac{1}{y+2}\,dy + \int \frac{-1}{y+7}\,dy =$$

$$= \log|y+2| - \log|y+7| + c = \log\left|\frac{y+2}{y+7}\right| + c$$

$$\int_0^{\log M} \frac{5}{y^2+9y+14}\,dy = \left[\log\frac{|y+2|}{|y+7|}\right]_0^{\log M} = \log\left(\frac{\log M+2}{\log M+7}\right) - \log\frac{2}{7}$$

$$\lim_{M\to+\infty} \int_1^M \frac{5}{x\left(14+9\log x+(\log x)^2\right)}\,dx = \lim_{M\to+\infty} \log\left(\frac{\log M+2}{\log M+7}\right) - \log\frac{2}{7} = \log\frac{7}{2}$$

$$\log 1 = 0$$

$$1 \leftarrow \frac{\cancel{\log M}\left(1+\frac{2}{\log M}\right)}{\cancel{\log(M)}\left(1+\frac{7}{\log M}\right)} \to 1$$

Es $\frac{1}{3x^2+7x+2} \sim \frac{1}{x^\alpha}$ se $\alpha \geq 1$

$\frac{1}{3x^2+7x+2} \sim \frac{1}{x}$ se $\alpha < 1$

per confronto asintotico ho che $\int_1^{+\infty} \frac{1}{3x^2+7x+2}\,dx < +\infty$
$\Updownarrow$
$\alpha > 1$

$\alpha = 2$ $\int_1^{+\infty} \frac{1}{3x^2+7x+2}\,dx = \lim_{M\to+\infty} \int_1^{M} \frac{1}{3x^2+7x+2}\,dx$

$3x^2+7x+2=0$ $\quad x = \frac{-7\pm\sqrt{49-24}}{6} = \frac{-7\pm5}{6} \left\langle \begin{array}{l} -2 \\ -\frac{1}{3} \end{array}\right.$

$\frac{1}{3x^2+7x+2} = \frac{1}{3}\left[\frac{A}{x+2} + \frac{B}{x+\frac{1}{3}}\right] = \frac{1}{3}\left[\frac{Ax+\frac{1}{3}A+Bx+2B}{(x+2)(x+\frac{1}{3})}\right]$

$$\begin{cases} A+B=0 \\ \frac{1}{3}A+2B=1 \end{cases} \quad \begin{cases} A=-B \\ 2B-\frac{1}{3}B=1 \end{cases} \quad \begin{cases} A=-B \\ \frac{5}{3}B=1 \end{cases} \quad \begin{cases} A=-\frac{3}{5} \\ B=\frac{3}{5} \end{cases}$$

$$\int \frac{1}{3x^2+7x+2}\,dx = \frac{1}{3}\int \frac{-\frac{3}{5}}{x+2}\,dx + \frac{1}{3}\int \frac{\frac{3}{5}}{x+\frac{1}{3}}\,dx =$$

$$= -\frac{1}{5}\lg|x+2| + \frac{1}{5}\lg\left|x+\frac{1}{3}\right| + c = \frac{1}{5}\lg\left|\frac{x+\frac{1}{3}}{x+2}\right| + c$$

$$\lim_{M\to+\infty}\int_1^M \frac{1}{3x^2+7x+2}\,dx = \lim_{M\to+\infty} \frac{1}{5}\cancel{\lg\left(\frac{M+\frac{1}{3}}{M+2}\right)} - \frac{1}{5}\lg\left(\frac{\frac{4}{3}}{3}\right) =$$

$$\lg\left(\frac{\cancel{M}\left(1+\frac{1}{3M}\right)}{\cancel{M}\left(1+\frac{2}{M}\right)}\right) \to \lg 1 = 0$$

$$= -\frac{1}{5}\lg\left(\frac{4}{9}\right) = \frac{1}{5}\lg\left(\frac{9}{4}\right)$$

4) $x^\alpha\left(1-\cos\frac{1}{x}\right) \underset{x\to+\infty}{\sim} x^\alpha \frac{1}{2x^2} \approx \frac{1}{x^{2-\alpha}}$ quindi converge $\iff 2-\alpha>1$ $\boxed{\alpha<1}$

$$\int_{\frac{2}{\pi}}^{+\infty} x^{-3}\left(1-\cos\frac{1}{x}\right)dx = \lim_{M\to+\infty}\int_{\frac{2}{\pi}}^{M} \frac{1}{x^3}\left(1-\cos\frac{1}{x}\right)dx$$

$$\int_{2/\pi}^{M} \frac{1}{x^3}\left(1-\cos\frac{1}{x}\right)dx = \int_{\pi/2}^{1/M} y^3\left(1-\cos y\right)\left(-\frac{1}{y^2}\right)dy = -\int_{\pi/2}^{1/M} y\,(1-\cos y)\,dy$$

$y=\frac{1}{x}$ $\quad x=\frac{1}{y}$ $\quad dx=-\frac{1}{y^2}dy$

$x=\frac{2}{\pi}\to y=\frac{\pi}{2}$

$x=M\to y=\frac{1}{M}$

$$= \int_{1/M}^{\pi/2} y(1-\cos y)\,dy$$

$$\int y(1-\cos y)\,dy = \text{per parti} = (y-\sin y)\cdot y - \int (y-\sin y)\,dy =$$

$f(y) = 1-\cos y \qquad F(y) = y - \sin y$

$g(y) = y \qquad g'(y) = 1$

$$= (y-\sin y)\,y - \frac{y^2}{2} - \cos y + c$$

$$\int_{1/M}^{\pi/2} y(1-\cos y)\,dy = \left[(y-\sin y)\,y - \frac{y^2}{2} - \cos y\right]_{\frac{1}{M}}^{\frac{\pi}{2}} =$$

$$= \left(\frac{\pi}{2} - \underbrace{\sin\frac{\pi}{2}}_{=1}\right)\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^2 \cdot \frac{1}{2} - \cancel{\cos\frac{\pi}{2}}^{=0} - \left(\frac{1}{M} - \sin\frac{1}{M}\right)\frac{1}{M} + \left(\frac{1}{M}\right)^2 \cdot \frac{1}{2} +$$

$$+ \cos\frac{1}{M} = \left(\frac{\pi}{2} - 1\right)\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \underbrace{\frac{1}{M}}_{\to 0}\underbrace{\left(\frac{1}{M} - \sin\frac{1}{M}\right)}_{\to 0} + \underbrace{\frac{1}{2}\left(\frac{1}{M}\right)^2}_{\to 0}$$

$$+ \cos\underbrace{\frac{1}{M}}_{\to 0} \xrightarrow[M\to+\infty]{} \left(\frac{\pi}{2} - 1\right)\frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2}\right)^2 + \underbrace{\cos 0}_{=1} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{\pi}{2} + 1$$

5) $\displaystyle\int_0^{+\infty} e^{-\sqrt{x}}\,dx = \lim_{M\to+\infty} \int_0^M e^{-\sqrt{x}}\,dx$

$$\int_0^M e^{-\sqrt{x}}\,dx \underset{y=\sqrt{x}}{=} \int_0^{\sqrt{M}} e^{-y}\cdot 2y\,dy = \text{per parti}$$

$x=y^2$
$dx = 2y\,dy$

$f'(y) = e^{-y} \to F(y) = -e^{-y}$
$g(y) = 2y \to g'(y) = 2$

$$= \left[-e^{-y}\cdot 2y\right]_0^{\sqrt{M}} - \int_0^{\sqrt{M}} -e^{-y}\cdot 2\,dy = -e^{-\sqrt{M}}\cdot 2\sqrt{M} + \cancel{e^0\cdot 0} + \left[-2e^{-y}\right]_0^{\sqrt{M}}$$

$$= -e^{-\sqrt{M}}\cdot 2\sqrt{M} - 2e^{-\sqrt{M}} + 2e^0 = e^{-\sqrt{M}}\cdot 2\sqrt{M} - 2e^{-\sqrt{M}} + 2$$

$$\lim_{M\to+\infty} \underbrace{e^{-\sqrt{M}}\,2\sqrt{M}}_{\downarrow\ 0} - \underbrace{2e^{-\sqrt{M}}}_{\downarrow\ e^{-\infty}=0} + 2 = 2$$

per confronto infiniti
$e^{-\sqrt{M}} \to 0 \quad \sqrt{M} \to +\infty$

6) $\int_{e}^{+\infty} \frac{\lg x}{x^3} dx = \lim_{M \to +\infty} \int_{e}^{M} \frac{\lg x}{x^3} dx$

$y = \lg x \qquad x = e^y \qquad dx = e^y dy$

$x^3 = e^{3y}$

$x = e \to y = \lg e = 1$

$x = M \to y = \lg M$

$\int_{e}^{M} \frac{\lg x}{x} dx = \int_{1}^{\lg M} \frac{y}{e^{3y}} \cdot e^y dy = \int_{1}^{\lg M} y \cdot e^{-2y} \; dy =$ per parti

$f(y) = e^{-2y} \to F(y) = -\frac{1}{2} e^{-2y}$

$g(y) = y \to g'(y) = 1$

$= \left[-\frac{1}{2} e^{-2y} \cdot y \right]_1^{\lg M} - \int_{1}^{\lg M} -\frac{1}{2} e^{-2y} dy = -\frac{1}{2} e^{-2 \lg M} \lg M + \frac{1}{2} e^{-2} +$

$+ \frac{1}{2} \left[-\frac{1}{2} e^{-2y} \right]_1^{\lg M} =$

$$= -\frac{1}{2} e^{-2\lg M} \lg M + \frac{1}{2} e^{-2} - \frac{1}{4} e^{-2\lg M} + \frac{1}{4} e^{-2} =$$

$$= -\frac{1}{2} \frac{1}{M^2} \lg M - \frac{1}{4} \frac{1}{M^2} + \frac{3}{4} e^{-2}$$

$$e^{-2\lg M} = \left(e^{\lg M}\right)^{-2} = M^{-2} = \frac{1}{M^2}$$

$$\lim_{M \to +\infty} -\frac{1}{2} \underbrace{\frac{1}{M^2} \lg M}_{\to 0} - \frac{1}{4} \underbrace{\frac{1}{M^2}}_{\to 0} + \frac{3}{4} e^{-2} = \frac{3}{4} e^{-2}$$

7

$$\frac{\sin\sqrt{x}}{x^\alpha} \sim \frac{\sqrt{x}}{x^\alpha} = \frac{1}{x^{\alpha-\frac{1}{2}}} \qquad \text{per } x \to 0$$

quindi l'integrale converge per confronto asintotico se e solo se $\alpha - \frac{1}{2} < 1$ $\quad \alpha < \frac{3}{2}$

$$\int_0^{\pi/2} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx \qquad \left\{ y = \sqrt{x} \right) \quad x = y^2 \quad dx = 2y\,dy \qquad = \int_0^{\sqrt{\pi/2}} \frac{\sin y}{\cancel{y}} \cdot 2\cancel{y}\, dy =$$

$$= \int_0^{\sqrt{\pi/2}} \sin y \,.\, 2\, dy = \left[-2\cos y\right]_0^{\sqrt{\pi/2}} = -2\cos\sqrt{\frac{\pi}{2}} + 2\cos 0$$

$$= -2\cos\sqrt{\frac{\pi}{2}} + 2$$