SOLUZIONI FOGLIO 7

1) $\sin x = x - \frac{x^3}{6} + o(x^3)$

N: $x \sin x - x^2 = x\left(x - \frac{x^3}{6} + o(x^3)\right) - x^2 = \cancel{x^2} - \frac{x^6}{6} + o(x^4) \cancel{- x^2} =$

$= x^4\left(-\frac{1}{6} + o(1)\right)$

$\cos x = 1 - \frac{x^2}{2} + o(x^2)$

D: $(1 - \cos x) \cdot x = \left(1 - 1 + \frac{x^2}{2} + o(x^2)\right) \cdot x = \left(\frac{x^2}{2} + o(x^2)\right) x =$

$= \frac{x^3}{2} + o(x^3) = x^3\left(\frac{1}{2} + o(1)\right)$

$$\lim_{x \to 0} \frac{x^4\left(-1/6 + o(1)\right)}{\cancel{x^3}\left(\frac{1}{2} + o(1)\right)} = 0$$

2) $\arctan x = x - \frac{x^3}{3} + o(x^3)$ $\nearrow$

N: $\arctan x - x = x - \frac{x^3}{3} + o(x^3) - x =$

$= x^3\left(-\frac{1}{3} + o(1)\right)$

$$\sin x^2 = (x^2) - \frac{1}{6}(x^2)^3 + o\left((x^2)^3\right) = x^2 - \frac{1}{6}x^6 + o(x^6)$$

$$x^\alpha - \sin x^2 = x^\alpha - x^2 + \frac{1}{6}x^6 + o(x^6)$$

$\boxed{\alpha < 2}$ $\quad$ $\alpha > 2$ $\quad$ $\boxed{\alpha = 2}$

$$x^\alpha\Big(1 - \overbrace{x^{2-\alpha}}^{\to\infty} + \overbrace{x^{6-\alpha}}^{\to 0} + \overbrace{o(x^{6-\alpha})}^{\to 0}\Big)$$

$$= x^\alpha(1 + o(1))$$

$$x^2 \cdot \Big(\underbrace{x^{\alpha-2}}_{\to 0} - 1 + \underbrace{\frac{1}{6}x^4 + o(x^4)}_{\to 0}\Big)$$

$$= x^2(-1 + o(1))$$

$$\frac{1}{6}x^6 + o(x^6) = x^6\left(\frac{1}{6} + o(1)\right)$$

1° caso $\quad \alpha < 2 \quad \lim_{x\to 0} \dfrac{x^3\left(-\frac{1}{3} + o(1)\right)}{x^\alpha(1 + o(1))} = 0 \quad$ perché $3 > 2 > \alpha$!

2° caso $\quad \alpha > 2 \quad \lim_{x\to 0} \dfrac{x^3\left(-\frac{1}{3} + o(1)\right)}{x^2(-1 + o(1))} = 0$

3° caso $\quad \alpha = 2 \quad \lim_{x\to 0^+} \dfrac{x^3\left(-\frac{1}{3} + o(1)\right)}{x^6\left(\frac{1}{6} + o(1)\right)} = \lim_{x\to 0^+} \underbrace{\frac{1}{x^3}}_{\to +\infty} \dfrac{\overbrace{\left(-\frac{1}{3} + o(1)\right)}^{\to -\frac{1}{3}}}{\underbrace{\left(\frac{1}{6} + o(1)\right)}_{\to \frac{1}{6}}} = -\infty$

3) $-\frac{1}{2n^2} \to 0$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$$x = -\frac{1}{2n^2} \to 0 \qquad e^{-1/2n^2} = 1 + \left(-\frac{1}{2n^2}\right) + \frac{1}{2}\left(-\frac{1}{2n^2}\right)^2 + o\left(-\frac{1}{2n^2}\right)^2 =$$

$$= 1 - \frac{1}{2n^2} + \frac{1}{8}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4) \qquad x = \frac{1}{n} \to 0$$

$$\cos\frac{1}{n} = 1 - \frac{1}{2}\left(\frac{1}{n}\right)^2 + \frac{1}{24}\left(\frac{1}{n}\right)^4 + o\left(\frac{1}{n}\right)^4 = 1 - \frac{1}{2}\frac{1}{n^2} + \frac{1}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

$$N: e^{-\frac{1}{2n^2}} - \cos\frac{1}{n} = \cancel{1} \cancel{-\frac{1}{2n^2}} + \frac{1}{8}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) \cancel{-1} \cancel{+\frac{1}{2}\frac{1}{n^2}} - \frac{1}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) =$$

$$= \frac{2}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) = \frac{1}{n^4}\left(\frac{1}{12} + o(1)\right)$$

$$D: \operatorname{arctg} x = x - \frac{x^3}{3} + o(x^3) \qquad x = \frac{1}{n} \qquad \operatorname{arctg}\frac{1}{n} = \frac{1}{n} - \frac{1}{3}\left(\frac{1}{n}\right)^3 + o\left(\frac{1}{n^3}\right)$$

$$\arctan\frac{1}{n}-\frac{1}{n}=\cancel{\frac{1}{n}}-\frac{1}{3}\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)-\cancel{\frac{1}{n}}=\frac{1}{n^3}\left(-\frac{1}{3}+o(1)\right)$$

~~ve~~ metto tutto insieme:

$$\frac{\frac{1}{n^4}\left(\frac{1}{12}+o(1)\right)}{n^{\alpha}\frac{1}{n^3}\left(-\frac{1}{3}+o(1)\right)}=\frac{n^{-4}}{n^{\alpha}n^{-3}}\cdot\frac{\left(\frac{1}{12}+o(1)\right)}{\left(-\frac{1}{3}+o(1)\right)}=$$

$$=\frac{1}{n^{\alpha-3+4}}\frac{\left(\frac{1}{12}+o(1)\right)}{\left(-\frac{1}{3}+o(1)\right)}=$$

$$=\frac{1}{n^{\alpha+1}}\frac{\left(\frac{1}{12}+o(1)\right)}{\left(-\frac{1}{3}+o(1)\right)}\longrightarrow$$

1) $\alpha+1=0$, $\alpha=-1$: $\dfrac{\frac{1}{12}}{-\frac{1}{3}}=-\dfrac{1}{4}$

2) $\alpha+1>0$, $\alpha>-1$: 0

3) $\alpha+1<0$, $\alpha<-1$: $-\infty$

(4) se $x \to +\infty$ $\quad \frac{1}{x} \to 0$

$$\log(1+t) \underset{t\to 0}{=} t - \frac{t^2}{2} + o(t)$$

$$t = \frac{1}{x} \to \log\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2}\left(\frac{1}{x}\right)^2 + o\left(\frac{1}{x}\right)^2$$

$$= \frac{1}{x} - \frac{1}{2}\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)$$

$$x^\alpha - x^2 \log\left(1+\frac{1}{x}\right) = x^\alpha - x^2\left[\frac{1}{x} - \frac{1}{2}\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right] =$$

$$= x^\alpha - x + \frac{1}{2} + o(1) \xrightarrow[x\to+\infty]{}$$

$\alpha > 1$

$$x^\alpha\Big(\underbrace{1 - \frac{x}{x^\alpha} + \frac{1}{2x^\alpha} + \frac{o(1)}{x^\alpha}}_{\to 1}\Big) \to +\infty$$

$\alpha < 1$

$$x\Big(\underbrace{\frac{x^\alpha}{x} - 1 + \frac{1}{2x} + \frac{o(1)}{x}}_{\to -1}\Big) \to -\infty$$

$\alpha = 1$

$$\frac{1}{2} + o(1) \to \frac{1}{2}$$

5) $\frac{1}{n^2} \to 0$ $\quad \sin x = x - \frac{x^3}{6} + o(x^3) \quad x = \frac{1}{n^2}$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + o(x^3) \quad x = \frac{1}{n^2}$$

$$\sin \frac{1}{n^2} - \operatorname{arctg} \frac{1}{n^2} = \frac{1}{n^2} - \frac{1}{6}\left(\frac{1}{n^2}\right)^3 + o\left(\frac{1}{n^2}\right)^3 - \left[\frac{1}{n^2} - \frac{1}{3}\left(\frac{1}{n^2}\right)^3 + o\left(\frac{1}{n^2}\right)^3\right] =$$

$$= \cancel{\frac{1}{n^2}} - \frac{1}{6}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right) - \cancel{\frac{1}{n^2}} + \frac{1}{3}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right) = \frac{1}{6}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right) =$$

$$= \frac{1}{n^6}\left(\frac{1}{6} + o(1)\right).$$

$$a_n = n^\alpha\left(\sin \frac{1}{n^2} - \operatorname{arctg} \frac{1}{n^2}\right) = n^\alpha \frac{1}{n^6}\left(\frac{1}{6} + o(1)\right) = \frac{1}{n^{6-\alpha}}\left(\frac{1}{6} + o(1)\right)$$

- $$\lim_n a_n = \lim_n \frac{1}{n^{6-\alpha}}\left(\frac{1}{6} + o(1)\right) = \begin{cases} \frac{1}{6} & \text{se } \alpha = 6 \\ 0 & \text{se } 6-\alpha > 0 \quad \alpha < 6 \\ +\infty & \text{se } 6-\alpha < 0 \quad \alpha > 6 \end{cases}$$

6) $\operatorname{tg} x = x + \frac{x^3}{3} + o(x^3)$ $\qquad x = \frac{1}{n^3} \to 0$ $\qquad e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$

$$\operatorname{tg} \frac{1}{n^3} = \frac{1}{n^3} + \frac{1}{3}\left(\frac{1}{n^3}\right)^3 + o\left(\frac{1}{n^3}\right)^3 = \frac{1}{n^3} + \frac{1}{3}\,\frac{1}{n^9} + o\left(\frac{1}{n^9}\right)$$

$$e^{+\frac{1}{n^3}} = 1 + \left(+\frac{1}{n^3}\right) + \frac{1}{2}\left(+\frac{1}{n^3}\right)^2 + \frac{1}{6}\left(+\frac{1}{n^3}\right)^3 + o\left(+\frac{1}{n^3}\right)^3 =$$

$$= 1 + \frac{1}{n^3} + \frac{1}{2}\,\frac{1}{n^6} + \frac{1}{6}\,\frac{1}{n^9} + o\left(\frac{1}{n^9}\right)$$

$$1 + \operatorname{tg}\left(\frac{1}{n^3}\right) - e^{1/n^3} = \cancel{1} + \cancel{\frac{1}{n^3}} + \frac{1}{3}\,\frac{1}{n^9} + o\left(\frac{1}{n^9}\right) - \cancel{1} - \cancel{\frac{1}{n^3}} - \frac{1}{2}\,\frac{1}{n^6} - \frac{1}{6}\,\frac{1}{n^9} + o\left(\frac{1}{n^9}\right)$$

$$= -\frac{1}{2}\,\frac{1}{n^6} + \frac{1}{6}\,\frac{1}{n^9} + o\left(\frac{1}{n^9}\right) = \frac{1}{n^6}\left[-\frac{1}{2} + \underbrace{o(1)}_{\frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)}\right]$$

$\frac{1}{n^\alpha} \to 0 \quad \alpha > 0!$

$$e^x = 1 + x + o(x)$$

$$e^{\frac{1}{n^\alpha}} - 1 = \cancel{1} + \frac{1}{n^\alpha} + o\left(\frac{1}{n^\alpha}\right) + o\left(\frac{1}{n^\alpha}\right) \cancel{-1} = \frac{1}{n^\alpha}(1 + o(1))$$

Metto tutto insieme

$$\frac{\frac{1}{n^6}\left(-\frac{1}{2} + o(1)\right)}{\frac{1}{n^\alpha}\left(1 + o(1)\right)} = \frac{n^{-6}}{n^{-\alpha}} \frac{\left(-\frac{1}{2} + o(1)\right)}{(1 + o(1))} = \frac{1}{n^{6-\alpha}} \frac{\left(-\frac{1}{2} + o(1)\right)}{(1 + o(1))} *$$

$$\lim_n \frac{1}{n^{6-\alpha}} \frac{\left(-\frac{1}{2} + o(1)\right)}{(1 + o(1))} = \begin{cases} \alpha = 6 & -\frac{1}{2} \\ 6 - \alpha > 0 \\ \alpha < 6 & 0 \\ 6 - \alpha < 0 \\ \alpha > 6 & -\infty \end{cases}$$

7) $\sin \alpha x = \alpha x - \frac{1}{6}(\alpha x)^3 + o(\alpha x)^3 = \alpha x - \frac{1}{6}\alpha^3 x^3 + o(x^3)$

$e^x = 1 + x + x^2/2 + x^3/6 + o(x^3)$

$$N: \; e^x - \sin\alpha x - 1 + x^3 = \cancel{1} + x + \frac{x^2}{2} + \frac{x^3}{6} - \alpha x + \frac{\alpha^3 x^3}{6} + o(x^3) - \cancel{1} + x^3 =$$

$\alpha = 1$ $\quad \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) = x^2\left(\frac{1}{2} + o(1)\right)$

$\alpha \neq 1$ $\quad (1-\alpha)\,x + \frac{x^2}{2} + \frac{x^3}{6}(1+\alpha^3) + o(x^3) =$

$= x\,(1-\alpha + o(1))$.

$\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{4!} + o(t^4) \qquad t = \sqrt{x} \to 0$

$\cos\sqrt{x} = 1 - \frac{(\sqrt{x})^2}{2} + o(\sqrt{x})^2 = 1 - \frac{x}{2} + o(x)$

$D: \; 1 - \cos\sqrt{x} = 1 - \cancel{1} + \frac{x}{2} + o(x) = \frac{x}{2} + o(x) = x\left(\frac{1}{2} + o(1)\right)$

$\boxed{\text{CASO } \alpha = 1}$ $\quad \lim_{x\to 0^+} \frac{x^{\cancel{2}}\left(\frac{1}{2} + o(1)\right)}{\cancel{x}\left(\frac{1}{2} + o(1)\right)} = 0$

$\boxed{\text{Caso } \alpha \neq 1}$ $\quad \lim_{x\to 0^+} \frac{\cancel{x}\,(1-\alpha + o(1))}{\cancel{x}\,(1/2 + o(1))} = \frac{1-\alpha}{1/2} = 2(1-\alpha).$

8) $\sqrt[3]{(1+x^2)} = (1+x^2)^{\frac{1}{3}}$

$$(1+t)^{\alpha} = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} t^2 + o(t^2)$$

$$t = x^2 \quad \alpha = \frac{1}{3}$$

$$(1+x^2)^{\frac{1}{3}} = 1 + \frac{1}{3}x^2 + \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)(x^2)^2 + o(x^2)^2 =$$

$$= 1 + \frac{1}{3}x^2 + \frac{1}{2}\cdot\frac{1}{3}\left(-\frac{2}{3}\right)x^4 + o(x^4) = 1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + o(x^4)$$

$$\sin x^2 = x^2 - \frac{1}{6}(x^2)^3 + o(x^2)^3 = x^2 - \frac{1}{6}x^6 + o(x^6)$$

$$\sqrt[3]{1+x^2} - 1 - \frac{1}{3}\sin x^2 = 1 + \frac{1}{3}x^2 - \frac{2}{9}x^4 + o(x^4) - 1 - \frac{1}{3}\left(x^2 - \frac{1}{6}x^6 + o(x^6)\right) =$$

$$= \cancel{1} \cancel{+\frac{1}{3}x^2} - \frac{2}{9}x^4 + o(x^4) \cancel{-1} \cancel{-\frac{1}{3}x^2} + \frac{1}{18}x^6 + o(x^6) =$$

$$\frac{1}{18}x^6 \in o(x^4)\,!$$

$$= -\frac{2}{9}x^4 + o(x^4) = x^4\left(-\frac{2}{9} + o(1)\right)$$

$$D: \quad 1-\cos\alpha x - x^2 = 1-\left(1-\frac{(\alpha x)^2}{2}+\frac{1}{24}(\alpha x)^4+o((\alpha x)^4)\right)-x^2=$$

$$=\cancel{1}-\cancel{1}+\frac{\alpha^2x^2}{2}+\frac{1}{24}\alpha^4x^4+o(x^4)-x^2=$$

$$=x^2\left(\frac{\alpha^2}{2}-1\right)+\frac{1}{24}\alpha^4x^4+o(x^4)$$

se $\alpha^2=2$ $(\alpha=\pm\sqrt{2})$ $\Rightarrow$ $\frac{1}{24}(2)^2\cdot x^4+o(x^4)=\frac{1}{6}x^4+o(x^4)$

$$=x^4\left(\frac{1}{6}+o(1)\right)$$

se $\alpha^2\neq 2$ $\quad \alpha\neq\pm\sqrt{2}$ $\Rightarrow$ $x^2\left[\frac{\alpha^2}{2}-1+o(1)\right]$

Metto insieme $\quad \alpha^2=2 \quad \frac{N}{D}=\frac{\cancel{x^4}\left(-\frac{2}{9}+o(1)\right)}{\cancel{x^4}\left(\frac{1}{6}+o(1)\right)}\longrightarrow\frac{-\frac{2}{9}}{\frac{1}{6}}=-\frac{2}{9}\cdot 6=-\frac{4}{3}$

$$\alpha^2\neq 2 \quad \frac{N}{D}\quad \frac{\cancel{x^4}^{\,2}\left(-\frac{2}{9}+o(1)\right)}{\cancel{x^2}\left(\frac{\alpha^2}{2}-1+o(1)\right)}\longrightarrow 0$$

9) $\lg(1+x) = x - \frac{x^2}{2} + o(x^2)$

$(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} t^2 + o(t^2)$

$\sqrt{1+2x} = (1+2x)^{1/2} =$

$\alpha = \frac{1}{2} \quad t = 2x$

$= 1 + \frac{1}{2} \cdot 2x + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)(2x)^2 + o(2x)^2 =$

$= 1 + x + \frac{1}{2} \cdot \frac{1}{2}\left(-\frac{1}{2}\right) \cdot 4x^2 + o(x^2) = 1 + x - \frac{1}{2}x^2 + o(x^2)$

$\lg(1+x) + 1 - \sqrt{1+2x} = x - \frac{x^2}{2} + o(x^2) + 1 - 1 - x + \frac{1}{2}x^2 + o(x^2) = o(x^2)$

non va bene! devo proseguire nello sviluppo di un altro termine!

$\lg(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$

$(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} t^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} t^3 + o(t^3)$

$\sqrt{1+2x} = 1 + x - \frac{1}{2}x^2 + \frac{1}{6}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(2x)^3 + o(x^3) =$

$$= 1 + x - \frac{1}{2}x^2 + \frac{1}{6}\cdot\left(\frac{1}{\cancel{2}}\right)\left(-\frac{1}{\cancel{2}}\right)\left(-\frac{3}{\cancel{2}}\right)\cancel{8}\,x^3 + o(x^3) =$$

$$= 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + o(x^3)$$

$$\lg(1+x) + 1 - \sqrt{1+2x} = \cancel{x} - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + o(x^3) + \cancel{1} - \cancel{1} - \cancel{x} + \cancel{\frac{1}{2}x^2} - \frac{1}{2}x^3 + o(x^3) =$$

$$= -\frac{1}{6}x^3 + o(x^3) = x^3\left(-\frac{1}{6} + o(1)\right)$$

D: $\sinh x = x + \frac{x^3}{6} + o(x^3)$

$$\lg(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$\sinh x - \lg(1+x) = \cancel{x} + \frac{x^3}{6} + o(x^3) \cancel{-x} + \frac{x^2}{2} - \frac{x^3}{3} + o(x^3) = \frac{x^2}{2} + o(x^2) =$$

$$= x^2\left(\frac{1}{2} + o(1)\right)$$

$$\frac{N}{D} = \frac{\cancel{x^3}\left(-\frac{1}{6}+o(1)\right)}{\cancel{x^2}\left(\frac{1}{2}+o(1)\right)} \to 0$$