RECAP :

controllo ottimo LQ di sistemi a tempo discreto:

caso a orizzonte finito $\quad t \in [0, T]$, $t, T \in \mathbb{N}$

$$u_T^\star(t) = \underset{t \in [0, T-1]}{\arg\min} \left(J_T(t) = x(T)^T S x(T) + \sum_{t=0}^{T-1} x(t)^T Q x(t) + u(t)^T R u(t) \right)$$

posto $\qquad S, Q \in \mathbb{R}^{n \times n}$ sdp

$$x(t+1) = F x(t) + G u(t) \qquad R \in \mathbb{R}^{m \times m} \text{ dp}$$
$$x(0) = x_0$$

▷ teorema principale

$$u_T^\star(t) = -K_T^\star(t)\, x(t) \quad \text{con} \quad K_T^\star(t) = \left(R + G^T M_T(t+1) G\right)^{-1} G^T M_T(t+1) F$$

dove $M_T(0), \dots, M_T(T)$ sequenze di matrici (simmetriche) sdp soluzione di ERD

$$\text{ERD} \quad \begin{cases} M(t) = F^T M(t+1) F - F^T M(t+1) G \left(R + G^T M(t+1) G\right)^{-1} G^T M(t+1) F \\ M(T) = S \end{cases}$$

$$\begin{tikzcd}
M_T(0) & M_T(1) \arrow[l] \arrow[d] & M_T(2) \arrow[l] \arrow[d] & \cdots \arrow[l] & M_T(T-1) \arrow[l] \arrow[d] & M_T(T) = S \arrow[l] \arrow[d] \\
& K_T^\star(0) \arrow[d] & K_T^\star(1) \arrow[d] & & K_T^\star(T-2) & K_T^\star(T-1) \\
& u_T^\star(0) \arrow[d] & u_T^\star(1) \arrow[d] & & & \\
x(0) \arrow[ur] & x(1) \arrow[ur] & x(2) & & &
\end{tikzcd}$$

controllo ottimo LQ di sistemi a tempo discreto :
caso a orizzonte infinito $t \in [0, +\infty)$ $t \in \mathbb{N}$

① modello di sistema da controllare

$$x(t+1) = F x(t) + G u(t) \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^m$$
$$x(0) = x_0$$

② funzionale costo da ottimizzare

$$J_\infty(t) = \sum_{t=0}^{+\infty} \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) \qquad \begin{matrix} Q \in \mathbb{R}^{n \times n} \\ R \in \mathbb{R}^{m \times m} \end{matrix} \; \begin{matrix} \text{sdp} \\ \text{dp} \end{matrix}$$

$$u^*_\infty(t) = \underset{t \in [0,+\infty)}{\operatorname{argmin}} \; J_\infty(t)$$

$$u^*_\infty(t) = \lim_{T \to \infty} u_T(t) = \lim_{T \to \infty} -\left((R + G^T M_T(t+1) G)^{-1} G^T M_T(t+1) F \right) x(t) \quad ?$$

$$\lim_{T \to \infty} M_T(t+1) = ?$$

condizioni necessarie e sufficienti per l'esistenza e l'unicità del limite

- (F, G) stabilizzabile
- (F, Q) rivelabile con $Q = H^T H$

↓

per ogni scelta della condizione finale $M_T(T)$ sdp, la soluzione $M_T(t)$ della ERD converge a un unico valore costante finito M_∞ quando $T \to \infty$. In particolare, M_∞ è l'unica soluzione sdp di

EARD $\quad M = F^T M F - F^T M G (R + G^T M G)^{-1} G^T M F + Q$

($\lim T \to \infty$ di ERD)

teorema principale

Per i sistemi a tempo discreto con (F, G) stabilizzabile e (F, Q) rivelabile, $Q = H^T H$, la legge di controllo ottimo è data da

$$u^*_\infty(t) = -K^*_\infty x(t) \quad \text{dove} \quad K^*_\infty = (R + G^T M_\infty G)^{-1} G^T M_\infty F$$

dove $M_\infty = M_\infty^T \in \mathbb{R}^{n \times n}$ unica soluzione sdp di EARD.

In corrispondenza all'ingresso $u^*_\infty(t)$, il funzionale costo assume il valore minimo

$$J^*_\infty = x_0^T M_\infty x_0$$

Proprietà stabilizzanti delle legge di controllo

proposizione

Sia $H \in \mathbb{R}^{p \times n}$ matrice tale che $Q = H^T H$.
Se (F, G) stabilizzante e (F, H) (equiv. (F, Q)) rivelabile
allora $u^*_\infty(t)$ rende il sistema in catena chiusa asintoticamente stabile

$A + F K^*_\infty \in \mathbb{R}^{n \times n}$: n autovalori all'interno del cerchio unitario

esempio

① modello di sistema

$$x(t+1) = F x(t) + g\, u(t) \qquad \text{con} \qquad F = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(0) = x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$(n = 2, m = 1)$

② funzionale costo

$$J_\infty(t) = \sum_{t=0}^{\infty} x(t)^T Q\, x(t) + r \cdot u^2(t) \qquad \begin{array}{l} r > 0 \\ Q \text{ sdp} \end{array}$$

A. $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad r = 1$

•) (F, g) stabilizzabile

$$R = \begin{bmatrix} g & Fg \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad : \quad \det R = 1 \neq 0$$

•) (F, Q) non rivelabile

$$\mathcal{O} = \begin{bmatrix} Q \\ QF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \quad : \quad \operatorname{rank} \mathcal{O} = 1 \Rightarrow \text{non osservabile}$$

$$PBH(z) = \begin{bmatrix} zI - F \\ Q \end{bmatrix} = \begin{bmatrix} z-2 & 0 \\ 0 & z-3 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

se $z = 2$ allora $\operatorname{rank} PBH(z) = 2$

se $z = 3$ allora $\operatorname{rank} PBH(z) = 1$

$\Rightarrow$ non rivelabile

posto $u(t) = -k^T x(t) = -\begin{bmatrix} k & 0 \end{bmatrix} x(t) = -\begin{bmatrix} k & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -k\, x_1(t)$
allora

$$x(t+1) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$x_1(t+1) = 2x_1(t) + u(t)$$
$$\overset{(}{=} \; 2x_1(t) - k x_1(t) = (2-k)\, x_1(t) \longrightarrow x_1(t) = (2-k)^t x_1(0)$$

$$J_\infty(t) = \sum_{t=0}^{\infty} x(t)^T Q\, x(t) + r u^2(t)$$

$$= \sum_{t=0}^{\infty} \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u^2(t)$$

$$= \sum_{t=0}^{\infty} x_1^2(t) + u^2(t)$$

$$= \sum_{t=0}^{\infty} x_1^2(t) + K^2 x_1^2(t)$$

$$= \sum_{t=0}^{\infty} (1+K)^2 x_1^2(t)$$

$$= (1+K)^2 \sum_{t=0}^{\infty} (2-K)^{2t} x_1^2(0)$$

$$= (1+K)^2 x_1^2(0) \sum_{t=0}^{\infty} (2-K)^{2t}$$

serie geometrica

$$\sum_{t=0}^{\infty} x^t = \frac{1}{1-x}$$

se $|x| < 1$

$$\boxed{|2-K| < 1 \quad \text{ovvero} \quad 1 < K < 3}$$

$$= (1+K)^2 x_1^2(0) \cdot \frac{1}{1-(2-K)^2}$$

$$= x_1^2(0) \cdot \frac{(1+K)^2}{-K^2+4K-3}$$

$$\frac{d}{dK} J_\infty(t) = 4\,\frac{K^2-K-1}{(-K^2+4K-3)^2} = 0 \quad \longleftrightarrow \quad K^2 - K - 1 = 0$$

$$K = \frac{1 \pm \sqrt{5}}{2}$$

$$K_\infty^* = \begin{bmatrix} K^* & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \end{bmatrix}$$

sistema retroazionato instabile

$$A = F - g K_\infty^* = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & 0 \\ \frac{-1-\sqrt{5}}{2} & 3 \end{bmatrix}$$

B. $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $r = 1$

- (F, g) stabilizzabile
- (F, Q) rivelabile

$$\Theta = \begin{bmatrix} Q \\ QF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 3 \end{bmatrix} \quad : \quad \text{rank}\,\Theta = 2 \;\Rightarrow\; \text{osservabile}$$

$$M_\infty = \text{idare}(F, g, Q, r, [\,], [\,]) = \begin{bmatrix} 126.245 & -196.030 \\ -196.030 & 316.144 \end{bmatrix}$$

$$K_\infty^* = (r + g^T M_\infty g)^{-1} g^T M_\infty F = \begin{bmatrix} -2.720 & 7.022 \end{bmatrix}^T$$

sistema retroazionato

$$A = F - gK_\infty^* = \begin{bmatrix} 4.720 & 7.022 \\ 2.720 & -4.022 \end{bmatrix} \longrightarrow \Lambda(A) = \{0.2728,\ 0.4252\}$$

☒

IMPLEMENTAZIONE DEL CONTROLLORE OTTIMO LQ

generalizzazione al caso con termini misti

$$\|y\|_P^2 = y^T P y$$

$$= (Hx + Ju)^T P (Hx + Ju)$$

$$= (u^T J^T + x^T H^T) P (Hx + Ju)$$

$$= x^T \boxed{H^T P H} x + u^T \boxed{J^T P J} u + u^T \boxed{J^T P H} x + x^T \boxed{H^T P J} u$$

$$= x^T Q x + u^T R u + u^T N^T x + x^T N u$$

$P^T = P \in \mathbb{R}^{p \times p}$

$y \in \mathbb{R}^p$

$y = Hx + Ju$

$$J_\infty(t) = \sum_{t=0}^{\infty} \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

$N \in \mathbb{R}^{n \times m}$

$$J_\infty(t) = \sum_{t=0}^{\infty} J'(t)$$

$$J'(t) = \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

$$= \boxed{x(t)^T Q\, x(t)} + u^T(t) R\, u(t) + x^T(t) N\, u(t) + u(t)^T N^T x(t)$$

$$\boxed{- x(t)^T N R^{-1} N^T x(t)} + x(t)^T N R^{-1} N^T x(t)$$

$$= x(t)^T (Q - N R^{-1} N^T)\, x(t) + \boxed{\begin{aligned} &+ u^T(t) R\, u(t) + x^T(t) N\, u(t) \\ &+ u(t)^T N^T x(t) + x(t)^T N R^{-1} N^T x(t) \end{aligned}}$$

completamento dei quadrati

dati $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $A = A^T \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times m}$ sdp
allora

$$y^T C y + x^T A B y + y^T B^T A x + x^T A B C^{-1} B^T A x = (y + C^{-1} B^T A x)^T C\, (y + C^{-1} B^T A x)$$

$x = x$, $y = u$, $A = I$, $B = N$, $C = R$

$$= x(t)^T \underbrace{\left(Q - NR^{-1}N^T \right)}_{\bar{Q}} x(t) + \underbrace{\left(u(t) + R^{-1}N^T x(t) \right)^T}_{\bar{u}(t)} R \left(u(t) + R^{-1}N^T x(t) \right)$$

① modello di sistema da controllare

$$x(t+1) = F x(t) + G u(t) \qquad\qquad u(t) = \bar{u}(t) - R^{-1}N^T x(t)$$

$$= F x(t) + G \bar{u}(t) - G R^{-1} N^T x(t)$$

$$= \left(F - G R^{-1} N^T \right) x(t) + G \bar{u}(t)$$

$$= \bar{F} x(t) + G \bar{u}(t) \qquad\qquad \bar{F} = F - G R^{-1} N^T$$

② funzionale costo da ottimizzare

$$J_\infty(t) = \sum_{t=0}^{\infty} J'(t)$$

$$= \sum_{t=0}^{\infty} x(t)^T \bar{Q} x(t) + \bar{u}(t)^T R \bar{u}(t) \qquad\qquad \bar{Q} = Q - NR^{-1}N^T$$

$$\bar{u}(t) = u(t) + R^{-1}N^T x(t)$$

se

- $\bar{Q}$ sdp
- $(\bar{F}, G)$ stabilizzabile
- $(\bar{F}, \bar{Q})$ rivelabile

allora

$$\bar{u}^\star_\infty(t) = -\bar{K}^\star_\infty x(t) \qquad \text{con} \qquad \bar{K}^\star_\infty = \left(R + G^T \bar{M}_\infty G \right)^{-1} G^T \bar{M}_\infty \bar{F}$$

dove $\bar{M}_\infty$ è soluzione di $\bar{M} = \bar{F}^T \bar{M} \bar{F} - \bar{F}^T \bar{M} G \left(R + G^T \bar{M} G \right)^{-1} G^T \bar{M} \bar{F} + \bar{Q}$

$$\bar{u}^\star_\infty(t) = u^\star_\infty(t) + R^{-1}N^T x(t)$$

$$u^\star_\infty(t) = \bar{u}^\star_\infty(t) - R^{-1}N^T x(t)$$

$$= -\left(R + G^T \bar{M}_\infty G \right)^{-1} G^T \bar{M}_\infty \bar{F} x(t) - R^{-1}N^T x(t)$$

$$= -\left(\left(R + G^T \bar{M}_\infty G \right)^{-1} G^T \bar{M}_\infty \bar{F} + R^{-1}N^T \right) x(t)$$

$$= -\left(\left(R + G^T \bar{M}_\infty G \right)^{-1} G^T M_\infty \left(F - GR^{-1}N^T \right) + R^{-1}N^T \right) x(t)$$

$$= -\left(R + G^T \bar{M}_\infty G \right)^{-1} \left(\underbrace{G^T \bar{M}_\infty \left(F - GR^{-1}N^T \right) + \left(R + G^T \bar{M}_\infty G \right) R^{-1}N^T}_{G^T \bar{M}_\infty F - \cancel{G^T \bar{M}_\infty G R^{-1} N^T} + N^T + \cancel{G^T \bar{M}_\infty G R^{-1} N^T}} \right) x(t)$$

$$= -\left(R + G^T \bar{M}_\infty G \right)^{-1} \left(G^T \bar{M}_\infty F + N^T \right) x(t)$$

$$u_\infty^\star(t) = -K_\infty^\star x(t) \qquad \text{con} \qquad K_\infty^\star = (R + G^T \bar{M}_\infty G)^{-1}(G^T \bar{M}_\infty F + N^T)$$

Scelta dei parametri del controllore

$Q, R \longrightarrow$ Q sdp
R dp
(F, Q) rivelabile

$Q = \text{diag}(q_i)$
$R = \text{diag}(r_i)$

$\hookrightarrow$ ① $Q = q \cdot I$
$R = r \cdot I$: gestione solo del rapporto q/r (cheap vs expensive control)

② regole euristica / regole di Bryson

$$q_i = \frac{1}{\bar{x}_i^2} \qquad \bar{x}_i = \max(x_i)$$
$$r_i = \frac{1}{\bar{u}_i^2} \qquad \bar{u}_i = \max(u_i)$$