RECAP :

controllo ottimo LQ di sistemi a tempo continuo nel caso a orizzonte finito

$$u_T^\star(t) = \underset{t\in[0,T]}{\arg\min}\ J_T(t) = x(T)^T S\, x(T) + \int_0^T x(t)^T Q\, x(t) + u(t)^T R\, u(t)\, dt$$

posto
$$\dot{x}(t) = Fx(t) + Gu(t)$$
$$x(0) = x_0$$

$\longrightarrow$ teorema principale

$$u_T^\star(t) = -K_T^\star(t)\,x(t) \qquad \text{con} \qquad K_T^\star(t) = R^{-1}G^T M_T(t)$$

$M_T(t) = M_T(t)^T$ sdp : unica soluzione di EDR

$$\begin{cases} -\dot{M}(t) = F^T M(t) + M(t)F - M(t)GR^{-1}G^T M(t) + Q \\ M(T) = S \end{cases}$$

esempio

$$u_T^\star(t) = -K_T^\star(t)\,x(t) \qquad \text{con} \qquad K_T^\star(t) = \frac{g}{r}\, m_T(t)$$

dove
$$m_T(t) = \begin{cases} \dfrac{1}{\frac{g^2}{r}(T-t) + \frac{1}{S}} & \text{se } f = q = 0 \\ m^+ + \dfrac{m^- - m^+}{1-\alpha e^{\beta(T-t)}} & \text{altrimenti} \end{cases}$$

$$m^+ = \frac{fr}{g^2} + \sqrt{\frac{f^2r^2}{g^4} + \frac{rq}{g^2}}$$

$$m^- = \frac{fr}{g^2} - \sqrt{\frac{f^2r^2}{g^4} + \frac{rq}{g^2}}$$

$$\beta = \frac{g^2}{r}(m^+ - m^-) \qquad\qquad \alpha = \frac{S-m^-}{S-m^+}$$

Soluzione dell' EDR

$M_T(t) \in \mathbb{R}^{n\times n}$

① sdp $\forall\, t \in [0,T]$

- •) $M_T(t) = M_T(t)^T$: struttura simmetrica di EDR
- •) $x^T(t) M_T(t) x(t) \geq 0 \;\forall\, x(t) \in \mathbb{R}^n$; $t \in [0,T]$

② limitata su $[0,T]$

$\exists\, \overline{M}$ tale che $M_T(t) \leq \overline{M} \quad \forall\, t \in [0,T]$

calcolo di $M_T(t)$

- (F,G) tempo-varianti $\longrightarrow$ soluzione numerica/algoritmica

  * from backward to forward integration

  $\tau = T - t$: $S(\tau) = M(T-\tau) \in \mathbb{R}^{n\times n}$

  backward integration

$$\begin{cases} -\dot{M}(t) = F^T M(t) + M(t)F - M(t)GR^{-1}G^T M(t) + Q \\ M(T) = S \end{cases}$$

  forward integration

$$\begin{cases} -\dot{S}(\tau) = F(\tau)^T S(\tau) + S(\tau)F(\tau) - S(\tau)G(\tau)R^{-1}G(\tau)^T S(\tau) + Q \\ S(0) = S \end{cases}$$

  * integrazione numerica : backward integration

- (F,G) tempo-invarianti $\longrightarrow$ soluzione algebrica

$\Rightarrow$ matrice hamiltoniana

$$\mathcal{H} = \begin{bmatrix} F & -GR^{-1}G^T \\ -Q & -F^T \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

proprietà spettrali

•) $\mathcal{H}$ è simile a $-\mathcal{H}^T$

$\exists\, T = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ tale che $T^{-1}\mathcal{H}T = -\mathcal{H}^T$

$\Lambda(\mathcal{H}) = \Lambda(-\mathcal{H})^T$

$\Rightarrow$ se $\lambda \in \Lambda(\mathcal{H})$ allora $-\lambda \in \Lambda(\mathcal{H})$

•) (F,G) stabilizzabile & (F,Q) rivelabile

$\Rightarrow$ se $\lambda \in \Lambda(H)$ allora $\lambda \in \mathbb{R}$

$\Rightarrow$ lo spettro della matrice hamiltoniana è reale e simmetrico rispetto all'asse immaginario

Proposizione

Dato il sistema (risolubile) retto dalla matrice $H \in \mathbb{R}^{2n \times 2n}$

$$\begin{bmatrix} \dot{Z}_1(t) \\ \dot{Z}_2(t) \end{bmatrix} = H \begin{bmatrix} Z_1(t) \\ Z_2(t) \end{bmatrix}, \quad \begin{bmatrix} Z_1(T) \\ Z_2(T) \end{bmatrix} = \begin{bmatrix} I_n \\ S \end{bmatrix}$$

$$Z_1(t), Z_2(t) \in \mathbb{R}^{n \times n}$$

allora l'unica soluzione sdp $M_T(t) \in \mathbb{R}^{n\times n}$ della EDR è data da

$$\begin{aligned} M_T(t) &= Z_2(t) Z_1(t)^{-1} \\ &= \left(W_{11} + W_{12} e^{D^+(t-T)} P e^{D^+(t-T)}\right) \left(W_{21} + W_{22} e^{D^+(t-T)} P e^{D^+(t-T)}\right)^{-1} \end{aligned}$$

dove

•) $W = \left[\begin{array}{c|c} W_{11} & W_{12} \\ W_{21} & W_{22} \end{array}\right] \in \mathbb{R}^{2n\times 2n}$ tale che

$$W^{-1} H W = \left[\begin{array}{c|c} D^- & 0 \\ 0 & D^+ \end{array}\right], \quad \begin{aligned} D^- &= \mathrm{diag}\{\lambda < 0, \lambda \in \Lambda(H)\} \\ D^+ &= \mathrm{diag}\{\lambda > 0, \lambda \in \Lambda(H)\} \\ &\overset{!}{=} -D^- \end{aligned}$$

•) $P = -(W_{22} - SW_{12})^{-1}(W_{21} - SW_{11}) \in \mathbb{R}^{n\times n}$

Dimostrazione

$M_T(t) = Z_2(t) Z_1(t)^{-1}$: soluzione della EDR

① $M_T(T) = S$

$Z_2(T) \cdot Z_1(T)^{-1} = S \cdot I_n^{-1} = S$ ✓

② $$\begin{aligned} \dot{M}_T(t) &= \frac{d}{dt}\left(Z_2(t) Z_1(t)^{-1}\right) \\ &= \left(\frac{d}{dt} Z_2(t)\right) \cdot Z_1(t)^{-1} + Z_2(t) \cdot \left(\frac{d}{dt} Z_1(t)^{-1}\right) \\ &= \dot{Z}_2(t) \cdot Z_1(t)^{-1} - Z_2(t) Z_1(t)^{-1} \dot{Z}_1(t) Z_1(t)^{-1} \end{aligned}$$

$$\frac{d}{dt} I = \frac{d}{dt}\left(Z(t) Z(t)^{-1}\right) = \left(\frac{d}{dt} Z(t)\right) \cdot Z(t)^{-1} + Z(t) \cdot \left(\frac{d}{dt} Z(t)^{-1}\right) = 0$$

$$\left(\frac{d}{dt} Z(t)^{-1}\right) = -Z(t)^{-1} \left(\frac{d}{dt} Z(t)\right) Z(t)^{-1}$$

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \mathcal{H} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} F & -GR^{-1}G^T \\ -Q & -F^T \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} F z_1 - GR^{-1}G^T z_2 \\ -Qz_1 - F^T z_2 \end{bmatrix}$$

$$\overset{!}{=} (-Qz_1 - F^T z_2) z_1^{-1} - z_2 z_1^{-1} (F z_1 - GR^{-1}G^T z_2) z_1^{-1}$$

$$\overset{!}{=} -Q - F^T z_2 z_1^{-1} - z_2 z_1^{-1} F + z_2 z_1^{-1} G R^{-1} G^T z_2 z_1^{-1}$$

$$\overset{!}{=} -Q - F^T M_T - M_T F + M_T G R^{-1} G^T M_T \qquad \checkmark$$

cambio di variabili

$$\begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = W^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad \boxed{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = W \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}}$$

$$\rightarrow \begin{cases} \begin{bmatrix} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \end{bmatrix} = \begin{bmatrix} D^- & 0 \\ 0 & D^+ \end{bmatrix} \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} \\ \begin{bmatrix} \bar{z}_1(T) \\ \bar{z}_2(T) \end{bmatrix} = W^{-1} \begin{bmatrix} I_n \\ S \end{bmatrix} \end{cases}$$

•) evoluzione dello stato

$$\boxed{\begin{aligned} \bar{z}_1(t) &= e^{D^-(t-T)} \bar{z}_1(T) \\ \bar{z}_2(t) &= e^{D^+(t-T)} \bar{z}_2(T) \end{aligned}}$$

•) stato finale

$$W \begin{bmatrix} \bar{z}_1(T) \\ \bar{z}_2(T) \end{bmatrix} = \begin{bmatrix} I_n \\ S \end{bmatrix} \rightarrow \begin{aligned} S W_{11} \bar{z}_1(T) + S W_{12} \bar{z}_2(T) &= I_n \cdot S \\ W_{21} \bar{z}_1(T) + W_{22} \bar{z}_2(T) &= S \end{aligned}$$

$$S W_{11} \bar{z}_1(T) + S W_{12} \bar{z}_2(T) = W_{21} \bar{z}_1(T) + W_{22} \bar{z}_2(T)$$

$$\Rightarrow \boxed{\begin{aligned} \bar{z}_2(T) &= -(W_{22} - S W_{12})^{-1} (W_{21} - S W_{11}) \bar{z}_1(T) \\ &\overset{!}{=} P \bar{z}_1(T) \end{aligned}}$$

$$\begin{aligned} z_1(t) &= W_{11} \bar{z}_1(t) + W_{12} \bar{z}_2(t) \\ &\overset{!}{=} W_{11} e^{D^-(t-T)} \bar{z}_1(T) + W_{12} e^{D^+(t-T)} P \bar{z}_1(T) \\ &\overset{!}{=} W_{11} e^{D^-(t-T)} \bar{z}_1(T) + W_{12} e^{D^+(t-T)} P e^{-D^-(t-T)} e^{D^-(t-T)} \bar{z}_1(T) \\ &\overset{!}{=} (W_{11} + W_{12} e^{D^+(t-T)} P e^{D^+(t-T)}) e^{D^-(t-T)} \bar{z}_1(T) \end{aligned}$$

$$z_2(t) = (W_{21} + W_{22} e^{D^+(t-T)} P e^{D^+(t-T)}) e^{D^-(t-T)} \bar{z}_1(T)$$

$$\begin{aligned} M_T(t) &= z_2(t) z_1^{-1}(t) = \\ &(W_{21} + W_{22} e^{D^+(t-T)} P e^{D^+(t-T)}) (W_{11} + W_{12} e^{D^+(t-T)} P e^{D^+(t-T)})^{-1} \end{aligned}$$

esempio

① modello di sistema

$$\dot{x}(t) = u(t) \qquad f = 0,\ g = 1$$
$$x(0) = 0$$

② funzionale costo

$$J_T(t) = \int_0^T x^2(t) + u^2(t)\ dt \qquad S = 0,\ q = 1,\ r = 1$$

▷ soluzione : teorema principale

$$u_T^*(t) = -k_T^*(t)\,x(t) \quad \text{con} \quad k_T^*(t) = \frac{g}{r} m_T(t) = m_T(t)$$

$$m_T(t) = m^+ + \frac{m^- - m^+}{1 - \alpha e^{\beta(T-t)}}$$

$$m^+ = \frac{fr}{g^2} + \sqrt{\frac{f^2 r^2}{g^4} + \frac{rq}{g^2}} = 0 + \sqrt{0+1} = 1$$

$$m^- = \frac{fr}{g^2} - \sqrt{\quad \dots \quad} = 0 - \sqrt{0+1} = -1$$

$$\beta = \frac{g^2}{r}(m^+ - m^-) = 2$$

$$\alpha = \frac{S - m^-}{S - m^+} = -1$$

$$m_T(t) = 1 + \frac{(-1) - 1}{1 - (-1)e^{2(T-t)}} = 1 - \frac{2}{1 + e^{2(T-t)}} = \frac{-1 + e^{2(T-t)}}{1 + e^{2(T-t)}}$$

▷ soluzione : matrice hamiltoniana

$$H = \begin{bmatrix} f & -\frac{g^2}{r} \\ -q & -f \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\det(\lambda I - H) = \det \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} = \lambda^2 - 1 \quad \begin{cases} \lambda = +1 = d^+ \\ \lambda = -1 = d^- \end{cases}$$

- autovettore relativo a $\lambda = -1$

$$Hx = -x$$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \longrightarrow \begin{matrix} -x_2 = -x_1 \\ -x_1 = -x_2 \end{matrix}$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix}$$

• autovettore relativo a $\lambda = 1$

$Hx = x$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \begin{matrix} -x_2 = x_1 \\ -x_1 = x_2 \end{matrix}$$

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \longrightarrow W^{-1} H W = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$m_T(t) = Z_2(t) Z_1(t)^{-1}$$
$$= \left(w_{21} + W_{22}\, e^{d^+(t-T)} p\, e^{d^+(t-T)} \right) \left(w_{11} + w_{12}\, e^{d^+(t-T)} p\, e^{d^+(t-T)} \right)^{-1}$$
$$= \left(1 - e^{2(t-T)}\right)\left(1 + e^{2(t-T)}\right)^{-1} = \frac{1 - e^{2(t-T)}}{1 + e^{2(t-T)}} \cdot \frac{e^{2(t-T)}}{e^{2(t-T)}}$$
$$= \frac{-1 + e^{2(t-T)}}{1 + e^{2(t-T)}}$$

$$p = -(w_{22} - S w_{12})^{-1}(w_{21} - S w_{11}) = -(1-0)^{-1}(1-0) = -1$$

$\boxtimes$

CONTROLLO OTTIMO LQ per SISTEMI a TEMPO CONTINUO nel CASO a ORIZZONTE INFINITO

① modello di sistema da controllare

$$\dot{x}(t) = F x(t) + G u(t) \qquad x(t) \in \mathbb{R}^n$$
$$x(0) = x_0 \qquad u(t) \in \mathbb{R}^m$$

② funzionale costo

$$J_\infty(t) = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt \qquad \begin{matrix} Q \in \mathbb{R}^{n \times n} & \text{sdp} \\ R \in \mathbb{R}^{m \times m} & \text{dp} \end{matrix}$$

$$u_\infty^*(t) = \operatorname{argmin}_{t \in [0, +\infty)} J_\infty(t)$$

il caso a orizzonte infinito coincide con il caso a orizzonte finito posto $T \to \infty$?

$$u_\infty^*(t) = \lim_{T \to \infty} u_T^*(t) = \lim_{T \to \infty} \left(-R^{-1} G^T M_T(t)\right) = -R^{-1} G^T \lim_{T \to \infty} M_T(t)$$

• esistenza
• unicità

Si

se (F, G) stabilizzabile e (F, H) rivelabile con $Q = H^T H$

$(\Longleftrightarrow (F, Q)$ rivelabile con $Q = H^T H)$

allora

$$\lim_{T \to \infty} M_T(t) = M_\infty \in \mathbb{R}^{n \times n} \quad \text{sdp}$$

unica soluzione di EAR

$$0 = F^T M + MF - MGR^{-1}G^T M + Q$$