RECAP

controllo ottimo : metodo sistematico per il calcolo della matrice K basato sulla soluzione dell'ottimizzazione di un funzionale

→ LQ

- Lineare : modelli di sistema lineari
- Quadratico : funzionali costo quadratici

•) sistemi a tempo continuo / discreto

•) ottimizzazione su orizzonte finito $t \in [0, T]$
su orizzonte infinito $t \in [0, +\infty)$

$$u^*(t) = \arg\min_t J(t) = J(u(t))$$

CONTROLLO OTTIMO LQ di SISTEMI a TEMPO CONTINUO a ORIZZONTE FINITO

① modello del sistema da controllare : sistema LTI a tempo continuo

$$\dot{x}(t) = Fx(t) + Gu(t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$
$$x(0) = x_0 \qquad t \in [0,T]$$

② funzionale costo da minimizzare : funzionale quadratico definito su $[0,T]$

$$J_T(t) = x(T)^T S\, x(T) + \int_0^T x(t)^T Q\, x(t) + u(t)^T R u(t)\, dt$$

$S, Q \in \mathbb{R}^{n\times n}$ sdp
$R \in \mathbb{R}^{m\times m}$ dp

$$\boxed{u_T^\star(t) = \arg\min_{t\in[0,T]} J_T(t)}$$

sia

•) $M(t) \in \mathbb{R}^{n\times n}$
$M(t) = M(t)^T$ differenziabile in $[0,T]$

•) $H(t) = \int_0^T \frac{d}{dt}\left(x(t)^T M(t)\, x(t)\right) dt$: funzione ausiliaria

$H(t)$

① regola fondamentale del calcolo integrale

$$H(t) = x(t)^T M(t)\, x(t) \Big|_0^T = x(T)^T M(T)\, x(T) - x(0)^T M(0)\, x(0)$$

② regole del calcolo differenziale + equazioni della dinamica

$$H(t) = \int_0^T \dot{x}(t)^T M(t)\, x(t) + x(t)^T \dot{M}(t)\, x(t) + x(t)^T M(t)\, \dot{x}(t)\, dt$$

$$= \int_0^T \left(Fx(t) + Gu(t)\right)^T M(t) x(t) + x(t)^T \dot{M}(t)\, x(t) + x(t)^T M(t)\left(Fx(t) + Gu(t)\right) dt$$

$$= \int_0^T x(t)^T F^T M(t) x(t) + u(t)^T G^T M(t) x(t) + x(t)^T \dot{M}(t)\, x(t) + x(t)^T M(t) F x(t) + x(t)^T M(t) G u(t)\, dt$$

$$= \int_0^T x(t)^T \left(F^T M(t) + \dot{M}(t) + M(t) F\right) x(t) + u(t)^T G^T M(t) x(t) + x(t)^T M(t) G u(t)\, dt$$

$$= \int_0^T \begin{bmatrix} u(t)^T & x(t)^T \end{bmatrix} \begin{bmatrix} 0_{m\times m} & G^T M(t) \\ M(t) G & F^T M(t) + \dot{M}(t) + M(t) F \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt$$

$$\Rightarrow x(T)^T M(T)\, x(T) - x(0)^T M(0)\, x(0) = \int_0^T \begin{bmatrix} u(t)^T & x(t)^T \end{bmatrix} \begin{bmatrix} 0 & G^T M(t) \\ M(t) G & F^T M(t) + \dot M(t) + M(t) F \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt$$

- $H'(t) = H(t)_{①} - H(t)_{②} = -H(t)_{①} + H(t)_{②} = 0$

$$H'(t) = x(0)^T M(0)\, x(0) - x(T)^T M(T)\, x(T) + \int_0^T \dots \, dt$$

$$J_T(t) = x(T)^T S\, x(T) + \int_0^T x(t)^T Q\, x(t) + u(t)^T R\, u(t)\, dt$$

$$= x(T)^T S\, x(T) + \int_0^T \begin{bmatrix} u^T(t) & x^T(t) \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt$$

$$J_T(t) = J_T(t) + H'(t)$$

$$= x(0)^T M(0)\, x(0) + x(T)^T \big(S - M(T) \big)\, x(T) + \int_0^T \begin{bmatrix} u(t)^T & x(t)^T \end{bmatrix} \begin{bmatrix} R & G^T M(t) \\ M(t) G & F^T M(t) + \dot M(t) + M(t) F + Q \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt$$

- EDR = Equazioni Differenziali di Riccati

$$\begin{cases} -\dot M(t) = F^T M(t) + M(t) F - M(t) G R^{-1} G^T M(t) + Q & (*) \\ M(T) = S & (*) \end{cases}$$

$\longrightarrow$ $M_T(t) \in \mathbb{R}^{n \times n}$: unica soluzione

$M_T(t) = M_T(t)^T$ sdp

$$J_T(t) = x(0)^T M_T(0)\, x(0) + x(T)^T \big(S - \underbrace{M_T(T)}_{0\ (*)} \big)\, x(T) + \int_0^T \begin{bmatrix} u(t)^T & x(t)^T \end{bmatrix} \begin{bmatrix} R & G^T M_T(t) \\ M_T(t) G & \boxed{M_T(t) G R^{-1} G^T M_T(t)} \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt$$

$$= x(0)^T M_T(0)\, x(0) + \int_0^T \begin{bmatrix} u(t)^T & x(t)^T \end{bmatrix} \begin{bmatrix} R \\ M_T(t) G \end{bmatrix} R^{-1} \begin{bmatrix} R & G^T M_T(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} dt$$

$R = R^T$

$M_T(t) = M_T(t)^T$

$$= x(0)^T M(0)\, x(0) + \int_0^T v(t)^T R^{-1} v(t)\, dt$$

R^{-1} dp

con $v(t) = \begin{bmatrix} R & G^T M_T(t) \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix}$

$\overset{!}{=} \underset{m \times m}{R} \underset{m \times 1}{u(t)} + G^T M_T(t) x(t) \in \mathbb{R}^{m \times 1}$: ingresso ausiliario

poiché
R^{-1} dp : $\left(x^T R^{-1} x > 0 \quad \forall x \neq 0 \right)$ allora $J_T(t)$ è minimizzato
quando

$v(t) = 0 \quad \longleftrightarrow \quad R u(t) + G^T M_T(t) x(t) = 0 \quad \longleftrightarrow \quad u(t) = R^{-1}\left(-G^T M_T(t)\right) x(t)$

$\overset{!}{=} -R^{-1} G^T M_T(t)\, x(t)$

$\overset{!}{=} -K(t)\, x(t)$

di conseguenza
$J_T(t) = x(0)^T M(0) x(0)$: minimo

$$\boxed{\begin{aligned} u_T^*(t) &= -K_T^*(t)\, x(t) \\ &\overset{!}{=} -\left(R^{-1} G^T M_T(t)\right) x(t) = \underset{t \in [0,T]}{\arg\min}\, J_T(t) \\ J_T(u_T^*(t)) &= J_T^*(t) = x(0)^T M(0)\, x(0) \end{aligned}}$$

TEOREMA

Per i sistemi a tempo continuo, la legge di controllo ottimo a orizzonte finito è data da

$$u_T^*(t) = -K_T^*(t)\, x(t) \qquad \text{con} \qquad K_T^* = R^{-1} G^T M_T(t)$$

dove $M_T(t) = M_T(t)^T \in \mathbb{R}^{n \times n}$ è l'unica soluzione sdp dell'EDR

$$\begin{cases} -\dot{M}(t) = F^T M(t) + M(t) F - M(t) G R^{-1} G^T M(t) + Q \\ M(T) = S \end{cases}$$

In corrispondenza all'ingresso $u_T^*(t)$, il funzionale costo assume il valore (minimo) $J_T^* = x(0)^T M_T(0)\, x(0)$

EDR eq. differenziali ordinarie quadratiche
$\rightarrow$ eq. HJB

esempio : caso scalare

① modello di sistema

$$\dot{x}(t) = f\,x(t) + g\,u(t) \qquad g \neq 0$$
$$x(0) = x_0$$

② funzionale costo

$$J_T(t) = x^T(T)\, s\, x(T) + \int_0^T q \cdot x^2(t) + r \cdot u^2(t)\, dt \qquad s, q \geq 0 \;,\; r > 0$$

$$\Rightarrow \quad u_T^*(t) = \arg\min_{t \in [0,T]} J_T(t)$$

teorema

$$u_T^*(t) = -K_T^*(t)\, x(t) \qquad \text{con} \quad K_T^*(t) = R^{-1} G^T M_T(t) \overset{!}{=} \frac{1}{r} \cdot g \cdot m_T(t)$$

dove $m_T(t)$ è soluzione di EDR

$$\begin{cases} -\dot{m}(t) = f m(t) + m(t) f - m(t) \cdot g \cdot \frac{1}{r} \cdot g\, m(t) + q \overset{!}{=} 2 f m(t) - \frac{g^2}{r} m^2(t) + q \\ m(T) = s \end{cases}$$

$\longrightarrow$ metodo di separazione delle variabili

$$\begin{cases} -\dot{m}(t) = 2 f m(t) - \frac{g^2}{r} m^2(t) + q \\ m(T) = s \end{cases}$$

$$-\dot{m}(t) = -\frac{dm(t)}{dt} = 2 f m(t) - \frac{g^2}{r} m^2(t) + q$$

$$\frac{1}{-2 f m(t) + \frac{g^2}{r} m^2(t) - q} \cdot dm(t) = dt$$

$$\int_{m(t)}^{m(T) = s} \frac{1}{-2 f m(t) + \frac{g^2}{r} m^2(t) - q} \cdot dm(t) = \int_t^T dt$$

• caso ① : $f = q = 0$

▷ $f = 0$: $\dot{x}(t) = g\,u(t)$

▷ $q = 0$: s

$$\int_{m(t)}^{s} \frac{1}{+\frac{g^2}{r} m^2(t)}\, dm(t) = \int_t^T dt \qquad \int \frac{1}{a x^2} dx = -\frac{1}{a} \cdot \frac{1}{x}$$

$$-\frac{1}{\frac{g^2}{r}}\cdot\frac{1}{m(t)}\Bigg|_{m(t)}^{S} = t\Bigg|_{t}^{T}$$

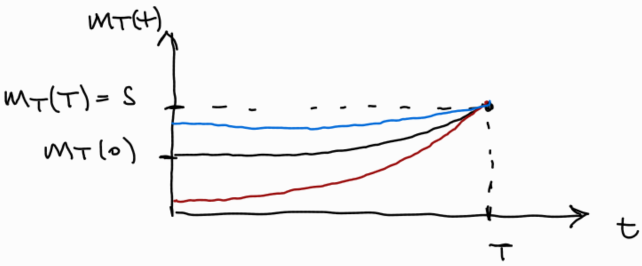
$$-\frac{r}{g^2}\cdot\frac{1}{S}+\frac{r}{g^2}\cdot\frac{1}{m(t)} = T-t \quad\Rightarrow\quad m_T(t) = \frac{1}{\frac{g^2}{r}(T-t)+\frac{1}{S}}$$

funzione monotona crescente di t

$$m_T(0) = \frac{1}{\frac{g^2}{r}T+\frac{1}{S}} \ : \quad \lim_{r\to 0^+} m_T(0) = 0 \quad \text{cheap control}$$

$$\lim_{r\to\infty} m_T(0) = S \quad \text{expensive control}$$

$$m_T(T) = S$$



$$k_T^*(t) = \frac{g}{r}\,m_T(t) = \frac{\frac{g}{r}}{\frac{g^2}{r}(T-t)+\frac{1}{S}}$$

- altrimenti $\quad f>0,\ q>0$; $f>0,\ q=0 \quad$ $f=0,\ q>0 \quad$ $f<0,\ q>0$; $f<0,\ q=0$

$$\int_{m(t)}^{S}\frac{1}{\frac{g^2}{r}m^2(t)-2fm(t)+q}\,dm(t) = \int_t^T dt$$

$$\frac{1}{\frac{g^2}{r}m^2(t)-2fm(t)+q} = \frac{1}{\frac{g^2}{r}\underbrace{\left(m^2(t)-\frac{2fr}{g^2}m(t)+\frac{qr}{g^2}\right)}}$$

$$m^+ = \frac{fr}{g^2}+\sqrt{\frac{f^2r^2}{g^4}+\frac{rq}{g^2}} \quad > 0$$

$$m^- = \frac{fr}{g^2}-\sqrt{\frac{f^2r^2}{g^4}+\frac{rq}{g^2}} \quad < 0$$

$$= \frac{1}{\frac{g^2}{r}(m(t)-m^+)(m(t)-m^-)}$$

$$\frac{1}{x-a}\cdot\frac{1}{x-b} = \frac{1}{a-b}\cdot\frac{a-b}{(x-a)(x-b)}$$

$$= \frac{1}{a-b} \cdot \frac{a-b+x-x}{(x-a)(x-b)}$$

$$= \frac{1}{a-b} \cdot \frac{(x-b)-(x-a)}{(x-a)(x-b)}$$

$$= \frac{1}{a-b} \left(\frac{1}{x-a} - \frac{1}{x-b} \right)$$

$x = m(t)$

$a = m^+$

$b = m^-$

$$= \underbrace{\frac{1}{\frac{g^2}{r}(m^+ - m^-)}}_{\frac{1}{\beta} > 0 \text{ poichè } \frac{g^2}{r} > 0 \text{ e } m^+ > m^-} \cdot \left(\frac{1}{m(t)-m^+} - \frac{1}{m(t)-m^-} \right)$$

$$\int_{m(t)}^{S} \frac{1}{\beta} \left(\frac{1}{m(t)-m^+} - \frac{1}{m(t)-m^-} \right) dm(t) = \int_t^T dt$$

$$\int \frac{1}{x}\, dx = \ln x$$

$$\frac{1}{\beta} \left(\ln(m(t)-m^+) - \ln(m(t)-m^-) \right) \Big|_{m(t)}^{S} = t \Big|_t^T$$

$$\log a - \log b = \log \frac{a}{b}$$

$$\frac{1}{\beta} \left(\ln\left(\frac{S-m^+}{S-m^-} \right) - \ln\left(\frac{m(t)-m^+}{m(t)-m^-} \right) \right) = T - t$$

$$\frac{1}{\beta} \cdot \ln \left(\underbrace{\frac{S-m^+}{S-m^-}}_{\frac{1}{\alpha}} \cdot \frac{m(t)-m^-}{m(t)-m^+} \right) = T - t$$

$$\ln \left(\frac{1}{\alpha} \cdot \frac{m(t)-m^-}{m(t)-m^+} \right) = \beta (T-t)$$

$$\frac{m(t)-m^-}{m(t)-m^+} = \alpha e^{\beta(T-t)}$$

$$1 - \frac{m^- - m^+}{m(t)-m^+} = \alpha e^{\beta(T-t)} \quad \Rightarrow \quad m_T(t) = m^+ + \frac{m^- - m^+}{1 - \alpha e^{\beta(T-t)}}$$

funzione monotona crescente in t
se $S > m^+$

$$K^*_T(t) = \frac{g}{r} m_T(t) = \frac{g}{r} \left(m^+ + \frac{m^- - m^+}{1 - \alpha e^{\beta(T-t)}} \right)$$

$\boxtimes$