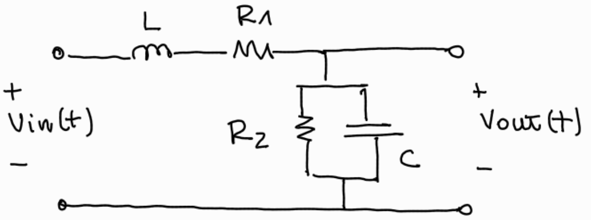
lez. 21 : simulazione delle prove in-itinere

① rete elettrica



$x_1(t) = i_L(t)$
$x_2(t) = V_C(t)$
$u(t) = V_{in}(t)$
$y(t) = V_{out}(t) = V_C(t)$

a) rappresentazione esterna

$$V_{Rj}(t) = R_j \, i_{Rj}(t) \qquad j = 1, 2$$
$$V_L(t) = L \cdot \frac{di_L(t)}{dt}$$
$$i_C(t) = C \cdot \frac{dV_C(t)}{dt}$$

$$i_L(t) = i_{R1}(t) = i_{R2}(t) + i_C(t) = \frac{V_C(t)}{R_2} + C \cdot \frac{dV_C(t)}{dt}$$

$$V_C(t) + V_{R1}(t) + V_L(t) = V_{in}(t) = V_C(t) + R_1 i_L(t) + L \frac{di_L(t)}{dt}$$

$$\begin{cases} i_L(t) = \dfrac{V_C(t)}{R_2} + C \cdot \dfrac{dV_C(t)}{dt} \\ V_{in}(t) = V_C(t) + R_1 i_L(t) + L \dfrac{di_L(t)}{dt} \end{cases}$$

$$\begin{aligned} V_{in}(t) &= V_C(t) + R_1\left(\frac{V_C(t)}{R_2} + C\frac{dV_C(t)}{dt}\right) + L \cdot \frac{d}{dt}\left(\frac{V_C(t)}{R_2} + C\frac{dV_C(t)}{dt}\right) \\ &= V_C(t) + \frac{R_1}{R_2} V_C(t) + R_1 C \frac{dV_C(t)}{dt} + \frac{L}{R_2}\frac{dV_C(t)}{dt} + LC \frac{d^2V_C(t)}{dt} \\ &= \left(1 + \frac{R_1}{R_2}\right) V_{out}(t) + \left(R_1 C + \frac{L}{R_2}\right)\frac{dV_{out}(t)}{dt} + LC\frac{d^2 V_{out}(t)}{dt} \end{aligned}$$

$$V_{in}(s) = \left(1 + \frac{R_1}{R_2}\right) V_{out}(s) + \left(R_1 C + \frac{L}{R_2}\right) s \cdot V_{out}(s) + LC\, s^2\, V_{out}(s)$$

$$G(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R_2}{(R_1 + R_2) + (R_1 R_2 C + L)\, s + R_2 L C\, s^2}$$

b) rappresentazione interna

$$\begin{cases} x_1(t) = \dfrac{x_2(t)}{R_2} + C\, \dot{x}_2(t) \\ V_{in}(t) = x_2(t) + R_1 x_1(t) + L\, \dot{x}_1(t) \end{cases}$$

$$\begin{cases} \dot{x}_2(t) = \dfrac{1}{C} x_1(t) - \dfrac{1}{R_2 C} x_2(t) \\ \dot{x}_1(t) = -\dfrac{R_1}{L} x_1(t) - \dfrac{1}{L} x_2(t) + \dfrac{1}{L} V_{in}(t) \end{cases}$$

$$V_{out}(t) = x_2(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} v_{in}(t)$$

$$V_{out}(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 0 \cdot V_{in}(t)$$

$$\begin{cases} \dot{x}(t) = F x(t) + g u(t) \\ y(t) = h^T x(t) + j u(t) \end{cases} \quad \text{con} \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \begin{matrix} u(t) = v_{in}(t) \\ y(t) = V_{out}(t) \end{matrix}$$

$$F = \begin{bmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{bmatrix}, \quad g = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}, \quad h^T = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad j = 0$$

2) sistema a tempo continuo non lineare

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), x_2(t), u(t)) = \cos x_2(t) - x_1(t) \cdot u(t) - 1 \\ \dot{x}_2(t) = f_2(x_1(t), x_2(t), u(t)) = \sin x_2(t) - x_1(t) \cdot u(t) \end{cases}$$

a) punti di equilibrio del sistema in corrispondenza all' ingresso $u(t) = \bar{u}$, $t \geq 0$

$[\cos\alpha \cos\beta - \sin\alpha \sin\beta = \cos(\alpha + \beta)]$

$$0 = \cos \bar{x}_2 - \bar{x}_1 \bar{u} - 1$$
$$0 = \sin \bar{x}_2 - \bar{x}_1 \bar{u}$$

se $\bar{u} = 0$
allora

$$\begin{cases} 0 = \cos \bar{x}_2 - 1 \\ 0 = \sin \bar{x}_2 \end{cases} \longrightarrow \bar{x}_2 = k\pi, \quad k \in \mathbb{Z}$$

$$\cos k\pi = 1 \longrightarrow k \text{ pari}$$

$$\Rightarrow (\bar{x}_1, \bar{x}_2) = (\alpha, 2k\pi) \qquad k \in \mathbb{Z} \qquad \alpha \in \mathbb{R}$$

se $\bar{u} \neq 0$
allora

$$0 = \cos \bar{x}_2 - \bar{x}_1 \bar{u} - 1$$
$$\bar{x}_1 = \frac{\sin \bar{x}_2}{\bar{u}} \longrightarrow 0 = \cos \bar{x}_2 - \sin \bar{x}_2 - 1$$

$$0 = \frac{\sqrt{2}}{2} \cos \bar{x}_2 - \frac{\sqrt{2}}{2} \sin \bar{x}_2 - \frac{\sqrt{2}}{2}$$

$$0 = \cos\left(\bar{x}_2 + \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}$$

$$\cos\left(\bar{x}_2 + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\bar{x}_2 + \frac{\pi}{4} = \pm \frac{\pi}{4} + 2k\pi \qquad k \in \mathbb{Z}$$

$$\begin{cases} \bar{x}_2' = -\frac{\pi}{4} + \frac{\pi}{4} + 2k\pi = 2k\pi \\ \bar{x}_2'' = -\frac{\pi}{4} - \frac{\pi}{4} + 2k\pi = -\frac{\pi}{2} + 2k\pi \end{cases}$$

$$\Rightarrow \quad (\bar{x}_1', \bar{x}_2') = (0, 2k\pi) \qquad k \in \mathbb{Z}$$
$$(\bar{x}_1'', \bar{x}_2'') = \left(-\frac{1}{\bar{u}}, -\frac{\pi}{2} + 2k\pi\right)$$

b) per $\bar{u} = 1$, modello linearizzato e stabilità dei punti di equilibrio

$$F = J_x^{(t)} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{(x_1,x_2,u)=(\bar{x}_1,\bar{x}_2,\bar{u})} = \begin{bmatrix} -\bar{u} & -\sin\bar{x}_2 \\ -\bar{u} & \cos\bar{x}_2 \end{bmatrix}$$

$$g = J_u^t = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}\right|_{(x_1,x_2,u)=(\bar{x}_1,\bar{x}_2,\bar{u})} = \begin{bmatrix} -\bar{x}_1 \\ -\bar{x}_1 \end{bmatrix}$$

•) $(\bar{x}_1, \bar{x}_2, \bar{u}) = (\bar{x}_1', \bar{x}_2', \bar{u}) = (0, 2k\pi, 1)$

$$F = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \qquad g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Delta_F(\lambda) = \det \begin{bmatrix} \lambda+1 & 0 \\ 1 & \lambda-1 \end{bmatrix} = (\lambda+1)(\lambda-1) = 0 \quad < \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 1 \end{matrix}$$

$\longrightarrow$ punto di equilibrio instabile

•) $(\bar{x}_1, \bar{x}_2, \bar{u}) = (\bar{x}_1'', \bar{x}_2'', \bar{u}) = \left(-1, -\frac{\pi}{2} + 2k\pi, 1\right)$

$$F = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \qquad g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Delta_F(\lambda) = \det \begin{bmatrix} \lambda+1 & -1 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + \lambda + 1 = 0 \quad < \begin{matrix} \lambda_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\ \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{matrix}$$

$\longrightarrow$ punto di equilibrio asintoticamente stabile

③ sistema a tempo continuo

$$\dot{x}(t) = F x(t) + g u(t) = \left[\begin{array}{cc|c} -1/2 & -1/2 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right] x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

a) forma di Jordan di F

$$F = \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \qquad \text{con} \qquad F_{11} = \begin{bmatrix} -1/2 & -1/2 \\ -1 & 0 \end{bmatrix}$$

$$F^J = \begin{bmatrix} F_{11}^J & 0 \\ 0 & F_{22}^J \end{bmatrix} \qquad F_{22} = [-1]$$

$$F^J_{22} = F_{22} = [-1]$$

$$F_{11} : \quad \Delta_{F_{11}}(\lambda) = \det \begin{bmatrix} \lambda + 1/2 & +1/2 \\ +1 & \lambda \end{bmatrix} = \lambda^2 + \frac{1}{2}\lambda - \frac{1}{2} = 0 < \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = \frac{1}{2} \end{matrix}$$

$$F^J_{11} = \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$F^J = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \hline 0 & 0 & -1 \end{array}\right]$$

b) evoluzione libera a partire da $x_0 = [0 \;\; 0 \;\; 1]^T$

$$x_\ell(t) = e^{Ft} \cdot x_0$$

$$e^{Ft} = T \cdot e^{F^J t} \, T^{-1}$$

$$F_{11} \cdot v_1 = \lambda_1 v_1 \qquad\qquad (F_{11} - \lambda_1 I) v_1 = 0$$

$$\begin{bmatrix} -1/2 & -1/2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_1^2 \end{bmatrix} = - \begin{bmatrix} v_1^1 \\ v_1^2 \end{bmatrix}$$

$$\begin{cases} -\frac{1}{2}(v_1^1 + v_1^2) = -v_1^1 \\ -v_1^1 = -v_1^2 \end{cases}$$

$$\begin{cases} v_1^1 = 1 \\ v_1^2 = 1 \end{cases}$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$F_{11} v_2 = \lambda_2 v_2$$

$$\begin{bmatrix} -1/2 & -1/2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2^1 \\ v_2^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} v_2^1 \\ v_2^2 \end{bmatrix}$$

$$\begin{cases} -\frac{1}{2}(v_2^1 + v_2^2) = \frac{1}{2} v_2^1 \\ -v_2^1 = \frac{1}{2} v_2^2 \end{cases}$$

$$\begin{cases} v_2^1 = 1 \\ v_2^2 = -2 \end{cases}$$

$$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$T = \begin{array}{c} \begin{matrix} \lambda_1 & \lambda_2 & \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \end{array} = \begin{bmatrix} T_{11} & \\ & T_{22} \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} T_{11}^{-1} & \\ & T_{22}^{-1} \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & -1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{Ft} = T e^{F^J t} T^{-1} = T \begin{bmatrix} e^{-t} & & \\ & e^{\frac{1}{2}t} & \\ & & e^{-t} \end{bmatrix} T^{-1} = \begin{bmatrix} \frac{2}{3}e^{-t} + \frac{1}{3}e^{\frac{1}{2}t} & \frac{1}{3}e^{-t} - \frac{1}{3}e^{\frac{1}{2}t} & 0 \\ \frac{2}{3}e^{-t} - \frac{2}{3}e^{\frac{1}{2}t} & \frac{1}{3}e^{-t} + \frac{2}{3}e^{\frac{1}{2}t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

$$x_e(t) = e^{Ft} x_0 = \begin{bmatrix} & & \\ & * & \\ & & \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^{-t} \end{bmatrix}$$

④ sistema a tempo discreto

$$x(t+1) = F x(t) + g u(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & -1 & 0 \\ -1/2 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

a) raggiungibilità e controllabilità del sistema indicando il numero di passi

$$R = \begin{bmatrix} g & Fg & F^2 g \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 3 & 1 \end{bmatrix} \quad : \quad \operatorname{rank} R = 2$$

$\Rightarrow$ Σ non raggiungibile

$$X_R = \operatorname{im} R = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \neq \mathbb{R}^3$$

$$\operatorname{im} F^3 \subseteq X_R$$

$$\operatorname{im} F^3 = \operatorname{im} \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & -1 & 0 \\ -1/2 & 2 & 1 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = X_R$$

$\Rightarrow$ Σ controllabile

$$\begin{aligned} X_C(1) &= \left\{ x \in \mathbb{R}^3 : Fx \in \operatorname{im} g \right\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 0 \\ \frac{1}{2}x_1 - x_2 \\ -\frac{1}{2}x_1 + 2x_2 + x_3 \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \right\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \frac{1}{2}x_1 - x_2 = -\frac{1}{2}x_1 + 2x_2 + x_3 \right\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_3 = x_1 - 3x_2 \right\} \\ &= \left\{ \underbrace{\begin{bmatrix} \alpha \\ \beta \\ \alpha - 3\beta \end{bmatrix}}_{\alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}}, \ \alpha, \beta \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\} \neq \mathbb{R}^3 \end{aligned}$$

$$X_c(2) = \{ \bar{x} \in \mathbb{R}^3 : F^2 x \in \operatorname{im}[g \;\; Fg] \}$$

$$F^2 = \begin{bmatrix} 0 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \boxed{\begin{bmatrix} 0 \\ -\frac{1}{2}x_1 + x_2 \\ \frac{1}{2}x_1 + x_3 \end{bmatrix}} \in \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\} = \boxed{\operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}} \right\}$$

$$\boxed{\left(-\frac{1}{2}x_1 + x_2\right) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \left(\frac{1}{2}x_1 + x_3\right) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$= \left\{ \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}, \ \alpha, \beta, \gamma \in \mathbb{R} \right\} = \mathbb{R}^3$$

$$\underbrace{\frac{1}{2}x_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{-\frac{1}{2}x_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}x_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} + \bar{x}_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Σ controllabile in 2 passi

b) ingresso di controllo che porti a zero lo stato iniziale $x_0 = [1 \;\; 1+a \;\; 1]^T$ nel minor numero di passi al variare di $a \in \mathbb{R}$

$$X_c(1) = \operatorname{span}\left\{ \begin{bmatrix} \cancel{1} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

$x_0 \in X_c(1)$ se $a = -1$

$$x(1) = F x(0) + g u(0)$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & -1 & 0 \\ -1/2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(0)$$

$$\Rightarrow u(0) = -\frac{1}{2}$$

$$X_c(2) = \mathbb{R}^3$$

$$x_0 \in X_c(2) \quad \forall a$$

$$x(2) = F^2 x_0 + [g \;\; Fg] \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1+a \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

$$\Rightarrow u(0) = \frac{a}{4} - \frac{1}{4}$$

$$u(1) = -\frac{3}{4}a - \frac{3}{4}$$