LCD (10/04/2024)

* Bisimilarity as a fix point

$\sim$ $\rightsquigarrow$ bisimulations + largest

fix point

→ theoretically "nice"

→ algorithmic consequences

→ assertion language

* recursion

$fact : \mathbb{N} \to \mathbb{N}$

$$fact(n) = \begin{cases} 1 & n = 0 \\ n * fact(n-1) & n > 0 \end{cases} \quad (*)$$

property of the function

$$\mathcal{F} = \{ f \mid f : \mathbb{N} \to \mathbb{N} \}$$

$$F : \mathcal{F} \to \mathcal{F}$$

$$F(f)(n) = \begin{cases} 1 & \text{if } n = 0 \\ n * f(n-1) & \text{if } n > 0 \end{cases}$$

we claim that the factorial function fact is such that

$$F(fact) = fact$$

i.e. fact is a fixpoint of F

↑ ? is there one fixpoint?

" " more than one?

Knaster - Tarski fixpoint theorem

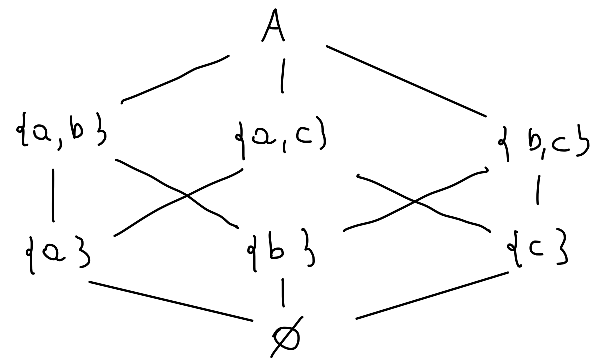
fixed points for monotone functions in complete lattices

SPECIAL CASE : powerset lattice

A set $\qquad 2^A = \{ X \mid X \subseteq A \}$

ordered by $\subseteq$

$A = \{a, b, c\}$



* **Monotone function**

$f : 2^A \to 2^A$ monotone if $\forall X, Y \in 2^A$

if $X \subseteq Y$ then $f(X) \subseteq f(Y)$

example :

$f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$

$f(X) = \{0\} \cup \{x+2 \mid x \in X\}$ monotone

$g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$

$g(X) = \{0\} \cup \{2 * x \mid x \notin X\}$ non monotone

$\emptyset \subseteq \{1\}$ $\qquad g(\emptyset) = \mathbb{P}$ $\qquad g(\{1\}) = \mathbb{P} \setminus \{2\}$

even numbers

<u>Knaster - Tarski</u> : Given $f: 2^A \to 2^A$ monotone. Then f has

1. largest fixpoint $\quad \mathrm{Fix}(f) = \bigcup \{ x \in 2^A \mid x \subseteq f(x) \}$

2. smallest fixpoint $\quad \mathrm{fix}(f) = \bigcap \{ x \in 2^A \mid f(x) \subseteq x \}$

<u>proof</u> (1) let $\quad \mathrm{Post} = \{ x \in 2^A \mid x \subseteq f(x) \}$

$$x_M = \bigcup \mathrm{Post}$$

↑ largest fixpoint of f

(i) fixpoint $\quad f(x_M) = x_M$

(ii) for all $x' \in 2^A$ if $x = f(x)$ then $x \subseteq x_M$

(i) $*$ $x_M \in \mathrm{Post}$ $\quad \boxed{x_M \subseteq f(x_M)}$ $(*)$

for all $x \in \mathrm{Post}$ $\quad x \subseteq \bigcup \mathrm{Post} = x_M$

$$\underset{x \in \mathrm{Post}}{x \subseteq f(x)} \underset{f \text{ is monotone}}{\subseteq} f(x_M)$$

$$\Downarrow$$

$$x_M = \bigcup \mathrm{Post} \subseteq f(x_M)$$

$*$ $f(x_M) \in \mathrm{Post}$

from $(*)$ and monotonicity of f

$$\underline{f(x_M)} \subseteq f(\underline{f(x_M)})$$

$$f(x_M) \subseteq \bigcup \mathrm{Post} = x_M \subseteq f(x_M)$$

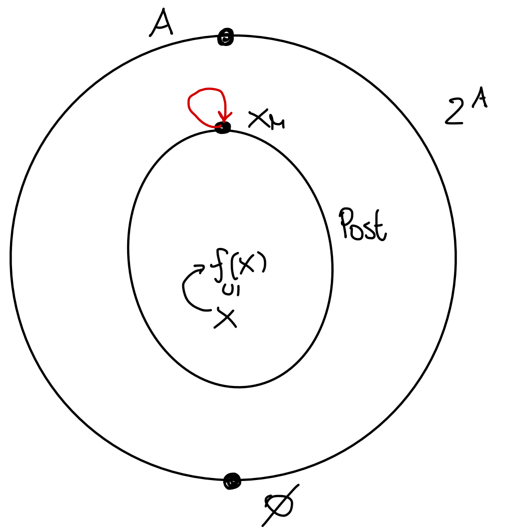
$$\Downarrow$$

$$x_M = f(x_M)$$

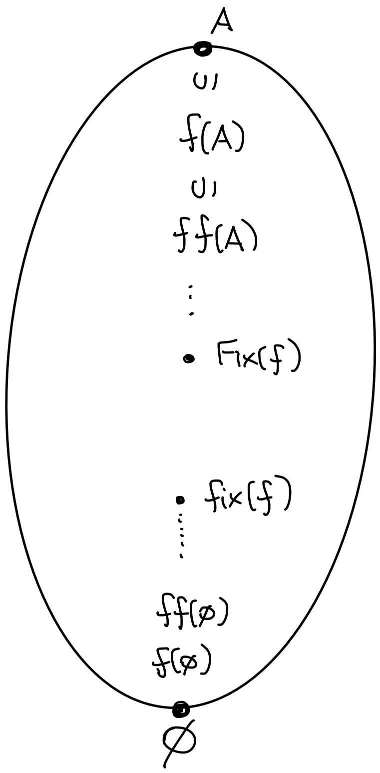
(ii) x_M is the largest fixpoint

for all $x \in 2^A$, if $f(x) = x$ then $x \in \mathrm{Post} \Rightarrow x \subseteq \bigcup \mathrm{Post} = x_M$

□



* If A is finite



$A \supseteq f(A)$

$f(A) \supseteq f(f(A))$

define $f^k(X)$ $\qquad f^0(X) = X \quad , \qquad f^{k+1}(X) = f\left(f^k(X)\right)$

OBSERVATION: If A is finite & $f: 2^A \to 2^A$ monotone

(a) $\mathrm{Fix}(f) = f^k(A)$ for some $k \in \mathbb{N}$

(b) $\mathrm{fix}(f) = f^n(\emptyset)$ for some $n \in \mathbb{N}$

EXERCISE for EXAM

* <u>Bisimilarity as a fixpoint</u>

bisimulation $\quad R \subseteq Proc \times Proc \quad R \in 2^{Proc \times Proc}$

if $\underbrace{P\ R\ Q}_{(P,Q) \in R}$ then (i) if $P \xrightarrow{\alpha} P'$ then $Q \xrightarrow{\alpha} Q'$ and $P'\ R\ Q'$

(ii) if $Q \xrightarrow{\alpha} Q'$ then $P \xrightarrow{\alpha} P'$ and $\underbrace{P'\ R\ Q'}_{(P',Q') \in R}$

we can consider $\quad F : 2^{Proc \times Proc} \to 2^{Proc \times Proc}$

$$F(R) = \left\{ (P,Q) \;\middle|\; \begin{array}{l} (i)\ \text{if } P \xrightarrow{\alpha} P' \text{ then } Q \xrightarrow{\alpha} Q' \text{ and } (P',Q') \in R \\ (ii)\ \text{if } Q \xrightarrow{\alpha} Q' \text{ then } P \xrightarrow{\alpha} P' \text{ and } (P',Q') \in R \end{array} \right\}$$

$\in 2^{Proc \times Proc}$

observe :

① Given $R \in 2^{Proc \times Proc}$, R is a bisimulation if

$$(P,Q) \in R \implies (P,Q) \in F(R)$$

i.e. $\quad R \subseteq F(R) \quad$ R is a post fixpoint of F

② F is monotone

③ $\sim \; = \bigcup \{ R \mid R \text{ bisimulation} \} = \bigcup \{ R \mid R \in Post \}$

$= Fix(F)$

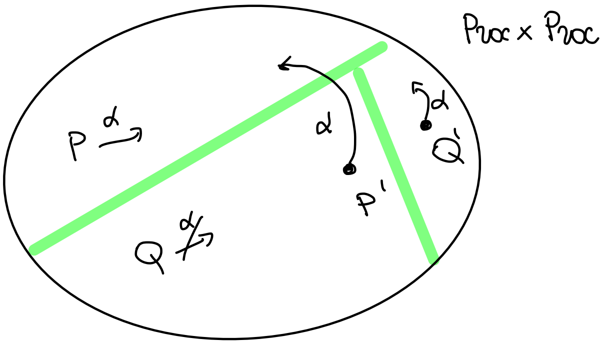
Knaster - Tarski

* If our processes are finite - state

$\sim \; = Fix(F) = F^k(Proc \times Proc)$

$Proc \times Proc \supsetneq F(Proc \times Proc) \supsetneq F(F(Proc \times Proc)) \supseteq \dots = \sim$

equivalences



$$F(R) = \left\{ (P,Q) \;\middle|\; \begin{array}{ll} (i) & \text{if } P \xrightarrow{\alpha} P' \text{ then } Q \xrightarrow{\alpha} Q' \text{ and } (P',Q') \in R \\ (ii) & \text{if } Q \xrightarrow{\alpha} Q' \text{ then } P \xrightarrow{\alpha} P' \text{ and } (P',Q') \in R \end{array} \right\}$$

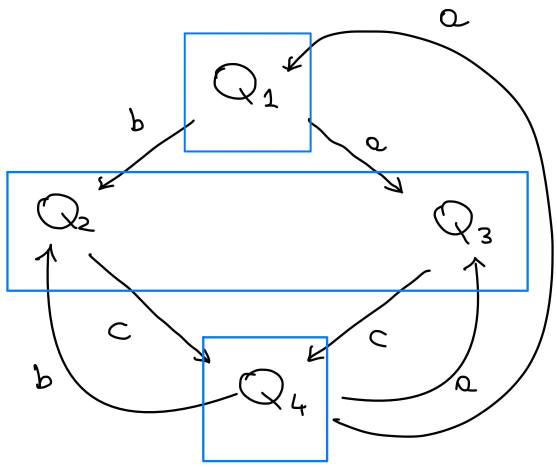
Example :

$Q_1 = b.Q_2 + a.Q_3$

$Q_2 = c.Q_4$

$Q_3 = c.Q_4$

$Q_4 = b.Q_2 + a.Q_3 + a.Q_1$



$Proc = \{Q_1, Q_2, Q_3, Q_4\}$

$F^0(Proc \times Proc) = Proc \times Proc$ $\qquad$ $I = \{(P,P) \mid P \in Proc\}$

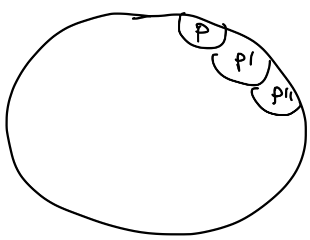
$F^1(Proc \times Proc) = \{ (Q_1,Q_4), (Q_4,Q_1), (Q_2,Q_3), (Q_3,Q_2) \} \cup I$

$F^2(Proc \times Proc) = \{ (Q_2,Q_3), (Q_3,Q_2) \} \cup I$

$F^3(Proc \times Proc) = F^2(Proc \times Proc) = \sim$

Complexity

$n = \#$ states $\qquad$ $m = \#$ transition



number of iterations $\leq n$

cost of each iteration m^2

$O(n\, m^2)$

Can you do better ?

$O(m\,n)$ Kannelakis & Smolka '83

$O(m \log n)$ Paige & Tarjan '87

data structures for partitions

- Trace equivalence

P-space complete

$NP \subseteq$ P-space complete $\subseteq$ EXP-TIME

Infinite states

- CCS bisimilarity is undecidable
- BPP (basic parallel processes)
  - → bisimilarity is decidable
  - → trace equivalence undecidable

⋮

* Petri nets

all equivalences are undecidable, reachability decidable (but not primitive recursive)