ESERCIZI su RAGGIUNGIBILITÀ e CONTROLLABILITÀ
RETROAZIONE dallo STATO

(1) $x(t+1) = Fx(t) + g\,u(t)$ $\qquad F = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} -2/3 \\ -1/3 \end{bmatrix}$

- raggiungibilità e controllabilità del sistema

$$R = \begin{bmatrix} g & Fg \end{bmatrix} = \begin{bmatrix} -2/3 & 5/3 \\ -1/3 & 4/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 & 5 \\ -1 & 4 \end{bmatrix}$$

$$\det R = \frac{1}{3^2}\left(-8+5\right) = -\frac{1}{9}\cdot 3 = -\frac{1}{3} \neq 0$$

$\Rightarrow$ Σ raggiungibile
Σ controllabile

(2) $x(t+1) = Fx(t) + g\,u(t)$ $\qquad F = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

- raggiungibilità e controllabilità del sistema

$$R = \begin{bmatrix} g & Fg & F^2g \end{bmatrix} = \begin{bmatrix} 0 & 3 & -3 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\det R = -3\left(-2-1\right) - 3\left(1+2\right) = +9-9 = 0$$

$\Rightarrow$ Σ non raggiungibile

autovalori di F

$$\Delta_F(\lambda) = \det(\lambda I - F) = \det\begin{bmatrix} \lambda+1 & -1 & -2 \\ 0 & \lambda+1 & 1 \\ 1 & 0 & \lambda-1 \end{bmatrix}$$

$$= (\lambda+1)\left(\lambda+1\right)\left(\lambda-1\right) + 1\,(-1) - 2\,(-\lambda-1)$$

$$= (\lambda+1)(\lambda^2-1) - 1 + 2\lambda + 2$$

$$= \lambda^3 - \lambda + \lambda^2 - 1 - 1 + 2\lambda + 2$$

$$= \lambda^3 + \lambda^2 + \lambda$$

$$= \lambda(\lambda^2+\lambda+1)$$

$\lambda_1 = 0$

$$\lambda_{2,3} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$PBH(z) = \begin{bmatrix} zI-F & g \end{bmatrix} = \begin{bmatrix} z+1 & -1 & -2 & 0 \\ 0 & z+1 & 1 & 1 \\ 1 & 0 & z-1 & 1 \end{bmatrix}$$

$$PBH(\lambda_{2,3}) = \begin{bmatrix} \frac{1}{2} \pm j\frac{\sqrt{3}}{2} & -1 & -2 & 0 \\ 0 & \frac{1}{2} \pm j\frac{\sqrt{3}}{2} & 1 & 1 \\ 1 & 0 & \frac{3}{2} \pm j\frac{\sqrt{3}}{2} & 1 \end{bmatrix} \rightarrow \text{rank} = 3$$

$\Rightarrow \Sigma$ controllabile

③ $x(t+1) = Fx(t) + Gu(t)$ $\qquad F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & \alpha & \alpha \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \alpha \in \mathbb{R}$

•) raggiungibilità e controllabilità del sistema al variare di $\alpha \in \mathbb{R}$

$$X_R(1) = \text{im}\, G = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \neq \mathbb{R}^3$$

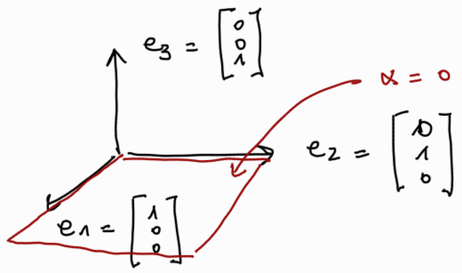
$$X_R(2) = \text{im}\left[G \;\; FG \right] = \text{im} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix}$$

$$= \begin{cases} \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} = X_R(1) & \text{se } \alpha = 0 \\ \mathbb{R}^3 & \text{se } \alpha \neq 0 \end{cases}$$

$$X_R(3) = \text{im}\left[G \;\; FG \;\; F^2G \right] = \text{im} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & -2 & -1+\alpha \\ 0 & 0 & 0 & \alpha & -\alpha & \alpha^2 \end{bmatrix}$$

$\Rightarrow \Sigma$ raggiungibile (in 2 passi) se $\alpha \neq 0$

NON raggiungibile se $\alpha = 0$



$$X_C(1) = \left\{ x \in \mathbb{R}^3 : Fx \in \text{im}\, G \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & \alpha & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \beta \\ \gamma \\ 0 \end{bmatrix}, \; \beta, \gamma \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \boxed{\begin{bmatrix} x_1 \\ -x_1 + x_2 + x_3 \\ \alpha(x_2 + x_3) \end{bmatrix} = \begin{bmatrix} \beta \\ \gamma \\ 0 \end{bmatrix}}, \; \beta, \gamma \in \mathbb{R} \right\}$$

$$= \left\{ \begin{array}{l} x_1 = \beta \\ -x_1 = \gamma \\ x_2 = -x_3 \end{array} \right. = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$x_1 = \beta$

$-x_1 + x_2 + x_3 = \gamma$

$\alpha(x_2 + x_3) = 0$

$\rightarrow x_2 = -x_3$

$$\beta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \beta \\ \delta \\ -\delta \end{bmatrix}$$

$$X_c(2) = \left\{ x \in \mathbb{R}^3 \;:\; F^2 x \in \operatorname{im}\begin{bmatrix} G & FG \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha & 1+\alpha \\ -\alpha & \alpha+\alpha^2 & \alpha+\alpha^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in X_e(2) \right\}$$

$$F^2 = F \cdot F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & \alpha & \alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & \alpha & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1+\alpha & 1+\alpha \\ -\alpha & \alpha+\alpha^2 & \alpha+\alpha^2 \end{bmatrix}$$

$$= \quad \alpha = 0$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \beta \\ \gamma \\ 0 \end{bmatrix}, \quad \beta, \gamma \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} x_1 \\ -2x_1 + x_2 + x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta \\ \gamma \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} \beta \\ \gamma \\ \delta \end{bmatrix} \right\} = \mathbb{R}^3$$

$\Rightarrow$ Σ controllabile (in 2 passi) $\forall\, \alpha$

- per $\alpha = 0$, determinare la forma canonica di raggiungibilità

$$F = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Sigma_R = \left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

$$R_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \qquad \operatorname{rank} R_R = 2$$

(4) $x(t+1) = Fx(t) + g\,u(t)$ $\qquad F = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

•) controllo (a energia minima) che porta il sistema da $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a $x(3) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

▷ esistenza

$$R = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \longrightarrow \operatorname{rank} R = 2 \Rightarrow \Sigma \text{ raggiungibile}$$

▷ calcolo

$$u(3) = R_3^T \left(R_3 R_3^T \right)^{-1} \left(x^* - F^3 x_0 \right)$$

$$R_3 = \begin{bmatrix} g & Fg & F^2g \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$F^3 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -1 \\ -7 \\ 8 \end{bmatrix} = \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

$$\mathcal{U} = \{ u' = u + \bar{u} \ , \ \bar{u} \in \ker R \} = u$$

$$\ker R_3 = \ker\left(\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \right) = \phi$$

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 0 \\ x_1 - x_2 &= 0 \quad \rightarrow \quad x_2 = 0 \end{aligned}$$

5) $x(t+1) = Fx(t) + g\,u(t)$ $\qquad F = \begin{bmatrix} 2 & 4 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, $g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- raggiungibilità e controllabilità del sistema

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix} \longrightarrow \operatorname{rank} R = 2$$

$\Rightarrow$ Σ non raggiungibile

autovalori di F

$$\Delta_F = \det \begin{bmatrix} \lambda-2 & -4 & 0 \\ 0 & \lambda+1 & 0 \\ -1 & -2 & \lambda-3 \end{bmatrix}$$

$$= (\lambda-2)(\lambda+1)(\lambda-3) \quad \begin{cases} \lambda = -1 \\ \lambda = 2 \\ \lambda = 3 \end{cases}$$

$$PBH(z) = \begin{bmatrix} z-2 & -4 & 0 & 1 \\ 0 & z+1 & 0 & 0 \\ -1 & -2 & z-3 & 0 \end{bmatrix}$$

$$PBH(\lambda=-1) \ : \ \text{rank} \begin{bmatrix} -3 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{bmatrix} = 2$$

$$PBH(\lambda=2) \ : \ \text{rank} \begin{bmatrix} 0 & -4 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{bmatrix} = 3$$

$$PBH(\lambda=3) \ : \ \text{rank} \begin{bmatrix} 1 & -4 & 0 & 1 \\ 0 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix} = 3$$

$\Rightarrow$ Σ non controllabile

•) forma canonica di raggiungibilità

$$T = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

x caso

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$F' = T^{-1} F T$$

$$\overset{!}{=} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & -6 & 0 \\ 1 & 5 & 2 \\ \hline 0 & 0 & -1 \end{array}\right]$$

$$G' = T^{-1} G$$

$$\overset{!}{=} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right]$$

$$\Sigma_R = \left(\begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$R_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \longrightarrow \quad \text{rank } R = 2$$

•) ingresso che porta il sistema da $x_0 = [0 \ 0 \ 0]^T$ a $x^* = [0 \ 0 \ 1]^T$ nel minor tempo possibile

▷ esistenza

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix} \qquad X_R = \text{im } R = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq \mathbb{R}^3$$

$x^* \in X_R$

▷ calcolo

$$t = 1 \ : \ X_R(1) = \text{im } g = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \not\ni x^*$$

$t = 2 \quad : \quad X_R(2) = \operatorname{im}\begin{bmatrix} g & Fg \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \ni x^*$

$$u_2 = R_2^T (R_2 R_2^T)^{-1} x^*$$

$$R_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_2 R^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$x^* = R_2 u_2$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \longrightarrow \begin{bmatrix} u(1) + 2u(0) \\ 0 \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{matrix} u(1) = -2 \\ u(0) = 1 \end{matrix} \quad \rightarrow \quad u_2 = \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(6) $x(t+1) = Fx(t) + g\,u(t) \qquad F = \begin{bmatrix} 0 & 0 & 1 \\ 1 & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \alpha \in \mathbb{R}$

•) controllore dead-beat al variare di α

$\Delta_A(\lambda) = p(\lambda) = \lambda^3 \quad (p_0 = p_1 = p_2 = 0) \qquad A = F + gK^T$

▷ esistenza

test PBH

$$\Delta_F(\lambda) = \det(\lambda I - F) = \det \begin{bmatrix} \lambda & 0 & -1 \\ -1 & \lambda - \alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda^2(\lambda - \alpha) = 0 \quad \begin{cases} \lambda_1 = 0 \\ \lambda_2 = \alpha \end{cases}$$

se $\alpha = 0$ allora $\lambda_1 = \lambda_2 = 0$ ...

se $\alpha \neq 0$ allora $\lambda_1 = 0 \quad m^a_1 = 2$; $\lambda_2 = \alpha \quad m^a_2 = 1$

$$PBH(z) = \begin{bmatrix} z & 0 & -1 & 1 \\ -1 & z - \alpha & 0 & 1 \\ 0 & 0 & z & 0 \end{bmatrix}$$

$$PBH(z = \alpha) = \begin{bmatrix} \alpha & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \end{bmatrix} : \operatorname{rank} \begin{cases} = 2 & \text{se } \alpha = -1 \\ = 3 & \text{se } \alpha \neq -1 \end{cases}$$

controllore dead-beat esiste $\longleftrightarrow$ $\alpha \neq -1$

▷ calcolo $K = [k_1 \;\; k_2 \;\; k_3]^T$

$$\Delta_A(\lambda) = \det(\lambda I - F - g k^T)$$

$$= \det\left(\begin{bmatrix} \lambda & 0 & -1 \\ -1 & \lambda-\alpha & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} [k_1 \;\; k_2 \;\; k_3] \right)$$

$$= \det\left(\begin{bmatrix} \lambda-k_1 & -k_2 & -1-k_3 \\ -1-k_1 & \lambda-\alpha-k_2 & -k_3 \\ 0 & 0 & \lambda \end{bmatrix} \right)$$

$$= \lambda^3 + (-k_1 - k_2 - \alpha)\lambda^2 + (k_1\alpha - k_2)\lambda$$

$$\begin{cases} -k_1 - k_2 - \alpha = 0 \\ k_1\alpha - k_2 = 0 \end{cases}$$

$$\begin{cases} k_1 = \dfrac{-\alpha}{1+\alpha} \\ k_2 = \dfrac{-\alpha^2}{1+\alpha} \\ k_3 = \beta \end{cases}$$