Lecture 6
Costs & Rewards

Alessandro Abate

Department of Computer Science
University of Oxford

Overview

- Specifying costs and rewards
  - DTMCs
  - hints at syntax for PRISM language
- Properties: expected reward values
  - instantaneous
  - cumulative
  - reachability
  - temporal logic extensions
- Model checking
  - computing reward values
- Case study
  - randomised contract signing

Costs and rewards

- We augment DTMCs with rewards (or, conversely, costs)
  - real-valued quantities assigned to states and/or transitions
  - these can have a wide range of possible interpretations
- Some examples:
  - elapsed time, power consumption, size of message queue, number of messages successfully delivered, net profit, …
- Costs or rewards?
  - mathematically, no distinction between rewards and costs
  - when interpreted, we assume that it is desirable to minimise costs and to maximise rewards
  - we will consistently use the terminology "rewards" regardless

Reward-based properties

- Properties of DTMCs augmented with rewards
  - allow a range of quantitative measures of the system: notion of expected value of rewards
  - (alternative reward structures possible, e.g., based on var)
  - rewards as specifications in an extension of PCTL

- More precisely, we use two distinct property classes:

- Instantaneous properties
  - e.g. the expected value of the reward at given time point

- Cumulative properties
  - e.g. the expected cumulated reward over a period/horizon

DTMC reward structures

- For a DTMC $(S,s_{init},\mathbf{P},L)$, a reward structure is a pair $(\underline{\rho},\iota)$
  - $\underline{\rho} : S \rightarrow \mathbb{R}_{\geq 0}$ is the state reward function (vector)
  - $\iota : S \times S \rightarrow \mathbb{R}_{\geq 0}$ is the transition reward function (matrix)

- Example (for use with instantaneous properties)
  - "size of message queue": $\underline{\rho}$ maps each state to the number of jobs in the queue, ι is not used (equal to zero everywhere)

- Examples (for use with cumulative properties)
  - "time-steps": $\underline{\rho}$ returns 1 for all states and ι is zero (equivalently, $\underline{\rho}$ is zero and ι returns 1 for all transitions)
  - "number of messages lost": $\underline{\rho}$ is zero and ι maps transitions corresponding to a message loss to 1
  - "power consumption": $\underline{\rho}$ is defined as the per-time-step energy consumption in each state and ι as the energy cost of each transition

Expected reward properties

- Expected ("average") values of rewards…
- Instantaneous
  - "the expected value of the state reward at time-step k"
  - e.g. "the expected nr. of jobs at exactly 90 seconds after start"
- Cumulative (time-bounded)
  - "the expected reward cumulated up to time-step k"
  - e.g. "the expected power consumption accrued over one hour"
- Reachability (also cumulative)
  - "the expected reward cumulated before reaching states $T \subseteq S$"
  - e.g. "the expected time for the algorithm to terminate"

Expectation

- Probability space (Ω, Σ, Pr)
  - probability measure $Pr : \Sigma \to [0,1]$

- Random variable X
  - a measurable function $X : \Omega \to \Delta$
  - usually real-valued, i.e.: $X : \Omega \to \mathbb{R}$

- Expected ("average") value of the random variable: Exp(X)

$$\mathrm{Exp}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$$

discrete case

$$\mathrm{Exp}(X) = \int_{\omega \in \Omega} X(\omega)\, d\Pr$$

Reachability + rewards

- Expected reward cumulated before reaching states $T \subseteq S$
- Define a random variable:
  - $X_{Reach(T)} : Path(s) \to \mathbb{R}_{\geq 0}$
  - where for an infinite path $\omega = s_0 s_1 s_2 \ldots$

$$X_{Reach(T)}(\omega) = \begin{cases} 0 & \text{if } s_0 \in T \\ \infty & \text{if } s_i \notin T \text{ for all } i \geq 0 \\ \sum_{i=0}^{k_T - 1} \underline{\rho}(s_i) + \iota(s_i, s_{i+1}) & \text{otherwise} \end{cases}$$

  - where $k_T = \min\{ j \mid s_j \in T \}$
- Then define:
  - $ExpReach(s, T) = Exp(s, X_{Reach(T)})$
  - denoting: expectation of the random variable $X_{Reach(T)}$ with respect to the probability measure Pr_s, i.e.:

$$\int_{\omega \in Path(s)} X_{Reach(T)}(\omega) \, dPr_s$$

Computing the rewards

- Determine states for which ProbReach(s, T) = 1

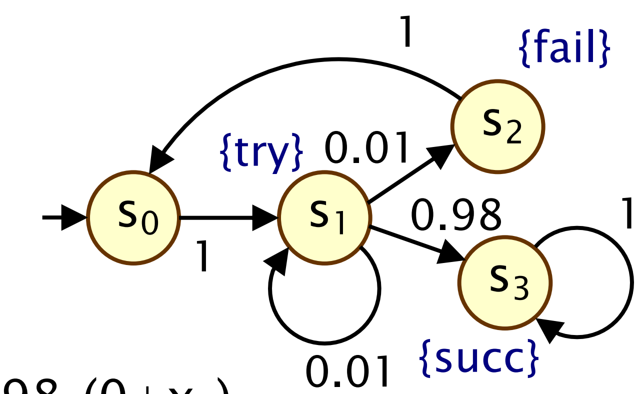
- Solve linear equation system:

  - ExpReach(s, T) =

$$
\begin{cases}
\infty & \text{if ProbReach(s, T)} < 1 \\
0 & \text{if } s \in T \\
\underline{\rho}(s) + \sum_{s' \in S} \mathbf{P}(s,s') \cdot \big(\iota(s,s') + \text{ExpReach}(s', T)\big) & \text{otherwise}
\end{cases}
$$

Example

- Let $\underline{\rho}$ = [0, 1, 0, 0] and $\iota(s,s') = 0$ for all $s,s' \in S$
- Compute ExpReach(s_0, {s_3})
  - ("expected number of times pass through s_1 to get to s_3")
- First check:
  - $\underline{\text{ProbReach}}(\{s_3\}) = \{ 1, 1, 1, 1 \}$
- Then solve linear equation system:
  - (letting x_i = ExpReach(s_i, {s_3})):
  - $x_0 = 0 + 1 \cdot (0 + x_1)$
  - $x_1 = 1 + 0.01 \cdot (0+x_2)+0.01 \cdot (0+x_1)+0.98 \cdot (0+x_3)$
  - $x_2 = 0 + 1 \cdot (0 + x_0)$
  - $x_3 = 0$
  - Solution: $\underline{\text{ExpReach}}(\{s_3\})$ = [100/98, 100/98, 100/98, 0]
- So: ExpReach(s_0, {s_3}) = 100/98 ≈ 1.020408



Specifying reward properties in PRISM

- PRISM extends PCTL to include expected reward properties
  - add an R operator, which is similar to the existing P operator

expected reward is ~r

  - $\phi \;::=\; \ldots \;|\; P_{\sim p}\,[\,\psi\,] \;|\; R_{\sim r}\,[\,I^{=k}\,] \;|\; R_{\sim r}\,[\,C^{\leq k}\,] \;|\; R_{\sim r}\,[\,F\,\phi\,]$

“instantaneous” “cumulative” “reachability”

  - where $r \in \mathbb{R}_{\geq 0}$, $\sim\, \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

- $R_{\sim r}\,[\,\cdot\,]$ means “the expected value of $\cdot$ satisfies $\sim r$”

Random variables for reward formulae

- Definition of random variables for the R operator:
  - for an infinite path $\omega = s_0 s_1 s_2 \ldots$

$$X_{I=k}(\omega) = \underline{\rho}(s_k)$$

$$X_{C\leq k}(\omega) = \begin{cases} 0 & \text{if } k = 0 \\ \sum_{i=0}^{k-1} \underline{\rho}(s_i) + \iota(s_i, s_{i+1}) & \text{otherwise} \end{cases}$$

$X_{F\phi}$
same as
$X_{Reach(Sat(\phi))}$
from earlier

$$X_{F\phi}(\omega) = \begin{cases} 0 & \text{if } s_0 \in Sat(\phi) \\ \infty & \text{if } s_i \notin Sat(\phi) \text{ for all } i \geq 0 \\ \sum_{i=0}^{k_\phi - 1} \underline{\rho}(s_i) + \iota(s_i, s_{i+1}) & \text{otherwise} \end{cases}$$

  - where $k_\phi = \min\{ j \mid s_j \vDash \phi \}$

Reward formula semantics

- Formal semantics of the three reward operators:

- For a state s in the DTMC:

  – $s \vDash R_{\sim r} [\ I^{=k}\ ] \Leftrightarrow Exp(s, X_{I=k}) \sim r$
  – $s \vDash R_{\sim r} [\ C^{\leq k}\ ] \Leftrightarrow Exp(s, X_{C\leq k}) \sim r$
  – $s \vDash R_{\sim r} [\ F\ \Phi\ ] \Leftrightarrow Exp(s, X_{F\Phi}) \sim r$

  $Exp(s, X_{F\Phi})$
  same as
  $ExpReach(s, Sat(\Phi))$
  seen earlier

  where: Exp(s, X) denotes the expectation of the random variable $X : Path(s) \rightarrow \mathbb{R}_{\geq 0}$ with respect to the probability measure Pr_s

- We can also define $R_{=?}$ [...] properties, as for the P operator
  – e.g. $R_{=?}$ [F Φ] returns the value $Exp(s, X_{F\Phi})$

Model checking reward operators

- As for model checking $P_{\sim p}$ [...], in order to check $R_{\sim r}$ [...]
  - compute reward values for all states, compare with bound r
- Instantaneous: $R_{\sim r}$ [$I^{=k}$] – compute Exp($X_{I=k}$)
  - solution of recursive equations
  - essentially: k matrix-vector multiplications
- Cumulative: $R_{\sim r}$ [$C^{\leq k}$] – compute Exp($X_{C\leq k}$)
  - solution of recursive equations
  - essentially: k matrix-vector multiplications
- Reachability: $R_{\sim r}$ [F ϕ] – compute Exp($X_{F\phi}$)
  - graph analysis + solution of linear system of equations
  - (see computation of ExpReach(s, T) earlier)

Model checking R operator has same complexity as P operator

Model checking $R_{\sim r}$ [$I^{=k}$]

- Expected instantaneous reward at step k
  - can be defined in terms of transient probabilities for step k

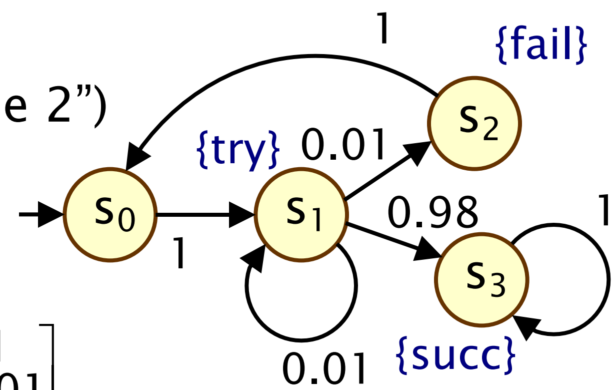
- $Exp(s, X_{I=k}) = \Sigma_{s' \in S} \, \pi_{s,k}(s') \cdot \underline{\rho}(s')$

- $\underline{Exp}(X_{I=k}) = \mathbf{P}^k \cdot \underline{\rho}$

- Yielding recursive definition:
  - $\underline{Exp}(X_{I=0}) = \underline{\rho}$
  - $\underline{Exp}(X_{I=k}) = \mathbf{P} \cdot \underline{Exp}(X_{I=(k-1)})$
  - i.e. k matrix-vector multiplications
  - note: “backward” computation (like bounded-until prob) rather than “forward” computation (like transient probs)

Example

- Let $\underline{\rho}$ = [0, 1, 0, 0] and $\iota(s,s') = 0$ for all $s,s' \in S$
- Compute $\text{Exp}(s_0, X_{I=2})$
  - ("probability of being in state s_1 at time 2")
  - $\text{Exp}(X_{I=0})$ = [0, 1, 0, 0]
  - $\text{Exp}(X_{I=1}) = \mathbf{P} \cdot \text{Exp}(X_{I=0})$



$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.01 \\ 0 \\ 0 \end{bmatrix}$$

  - $\text{Exp}(X_{I=2}) = \mathbf{P} \cdot \text{Exp}(X_{I=1})$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0.01 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.0001 \\ 1 \\ 0 \end{bmatrix}$$

- Result: $\text{Exp}(s_0, X_{I=2}) = 0.01$

Model checking $R_{\sim r}$ [$C^{\leq k}$]

- Expected reward cumulated up to time step k

- Again, a recursive definition:

$$\mathrm{Exp}(s,\ X_{C\leq k}) \ = \begin{cases} 0 & \text{if } k=0 \\ \underline{\rho}(s) + \sum_{s'\in S} \mathbf{P}(s,s') \cdot \left(\boldsymbol{\iota}(s,s') + \mathrm{Exp}(s',\ X_{C\leq k-1})\right) & \text{if } k>0 \end{cases}$$

- And in matrix/vector notation:

$$\underline{\mathrm{Exp}}(X_{C\leq k}) \ = \begin{cases} 0 & \text{if } k=0 \\ \underline{\rho} + (\mathbf{P} \bullet \boldsymbol{\iota}) \cdot \underline{1} + \mathbf{P} \cdot \underline{\mathrm{Exp}}(X_{C\leq k-1}) & \text{if } k>0 \end{cases}$$

  - where $\bullet$ denotes Schur (pointwise) matrix multiplication
  - and $\underline{1}$ is a unit vector (of all 1s)

Case study: Contract signing

- Two parties want to agree on a contract
  - each will sign if the other will sign, but do not trust each other
  - there may be a trusted third party (judge)
  - but it should only be used if something goes wrong
- In real life: contract signing with pen and paper
  - sit down and write signatures simultaneously
- On the Internet…
  - how to exchange commitments on an asynchronous network?
  - "partial secret exchange protocol" [EGL85]

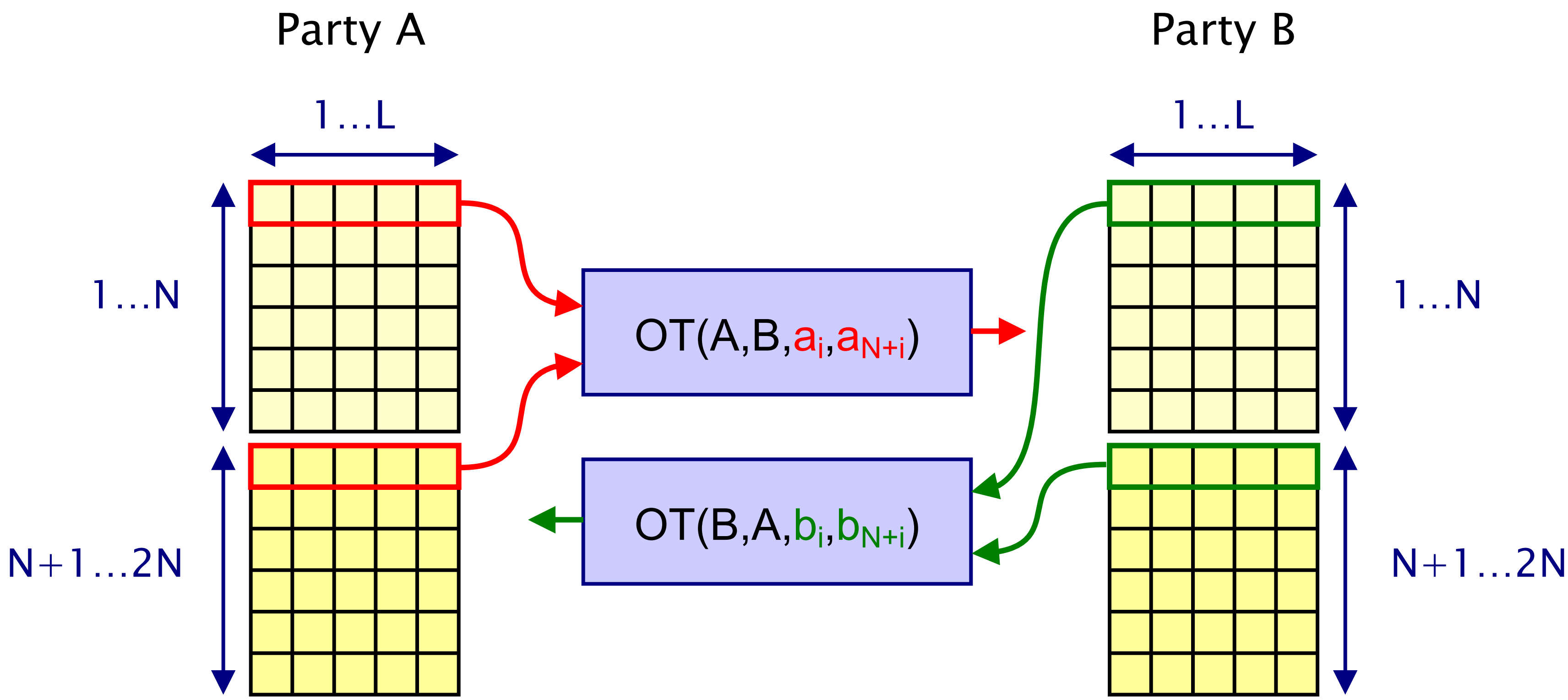
Contract signing – EGL protocol

- Partial secret exchange protocol for 2 parties (A and B)

- A (B) holds 2N secrets $a_1,\ldots,a_{2N}$ $(b_1,\ldots,b_{2N})$
  - a secret is a binary string of length L
  - secrets partitioned into pairs: e.g. $\{ (a_i, a_{N+i}) \mid i=1,\ldots,N \}$
  - A (B) committed if B (A) knows one of A's (B's) pairs

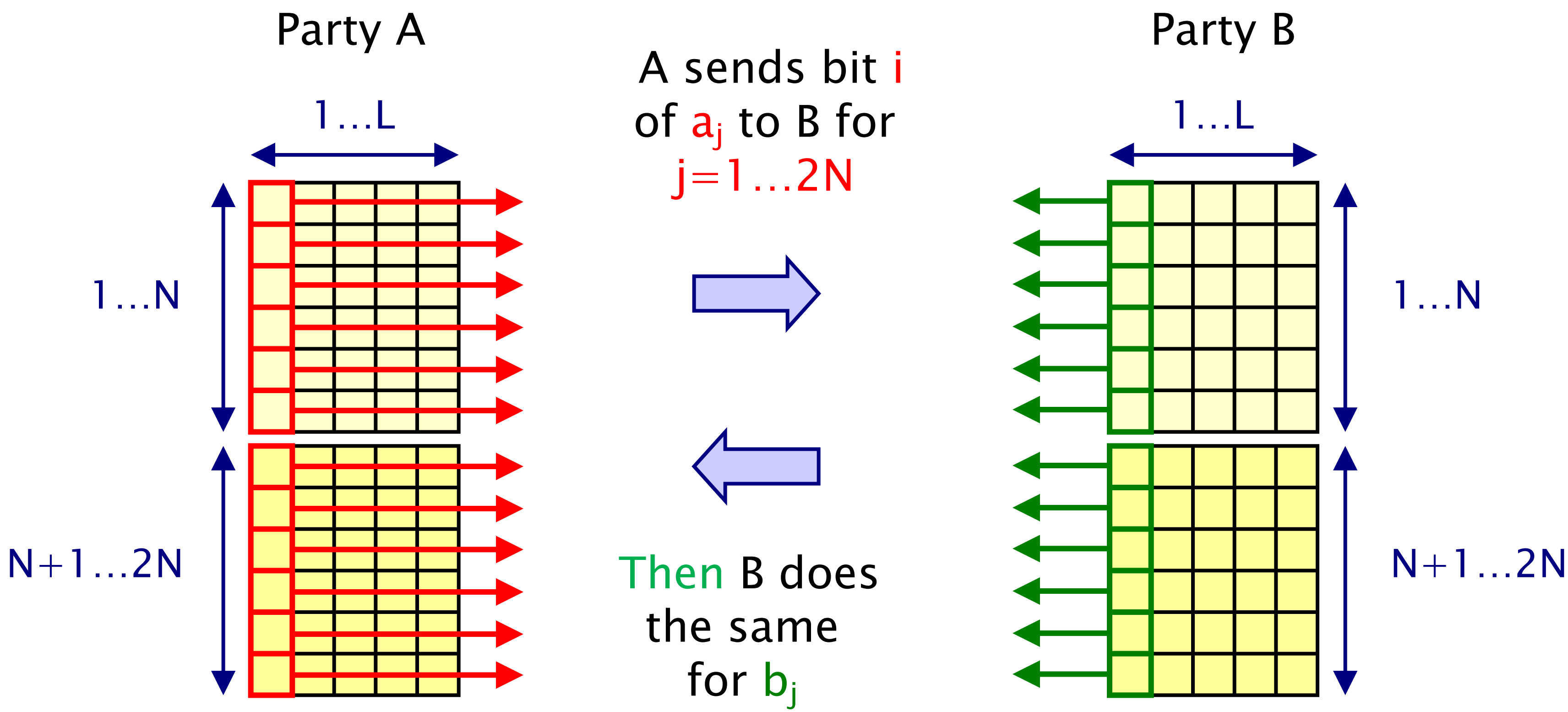
- Uses "1-out-of-2 oblivious transfer protocol" OT(S,R,x,y)
  - Sender S sends x and y to receiver R
  - R receives x with probability ½ otherwise receives y
  - S does not know which one R receives
  - if S cheats then R can detect this with probability ½

EGL protocol – Step 1



(repeat for i=1…N)

EGL protocol – Step 2



(repeat over i=1…L)

Contract signing – Results

- Modelled in PRISM as a DTMC (no concurrency) [NS06]

- Highlights a weakness in the protocol
  - party B can act maliciously by quitting the protocol early
  - this behaviour not considered in the original analysis

- PRISM analysis shows
  - if B stops participating in the protocol as soon as he/she has obtained one of A pairs, then, with probability 1, at this point:
    - B possesses a pair of A's secrets
    - A does not have complete knowledge of any pair of B's secrets
  - protocol is not fair under this attack:
  - B has a distinct advantage over A

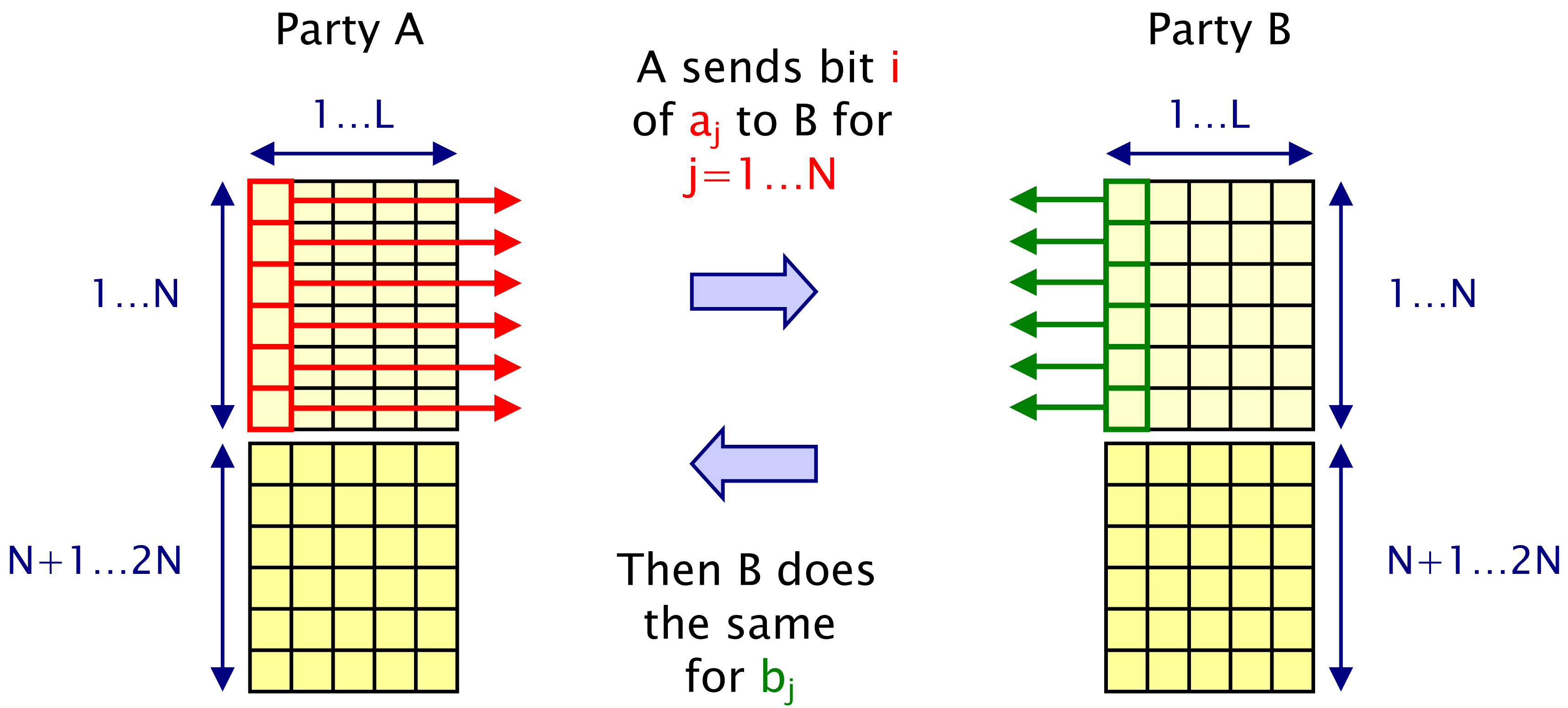
Contract signing – Results

- The protocol is unfair because in step 2:
  - A sends a bit for each of its secrets before B does
- Can we make this protocol fair by changing the message sequence scheme?
- Since the protocol is asynchronous the best we can hope for is:
  - B (or A) has this advantage with probability ½
- We consider 3 possible alternative message sequence schemes (EGL2, EGL3, EGL4)

Contract signing – EGL2

(step 1)
…
(step 2)
for (i=1,…,L)
  for (j=1,…,N) A transmits bit i of secret a_j to B
  for (j=1,…,N) B transmits bit i of secret b_j to A
  for (j=N+1,…,2N) A transmits bit i of secret a_j to B
  for (j=N+1,…,2N) B transmits bit i of secret b_j to A

Modified step 2 for EGL2

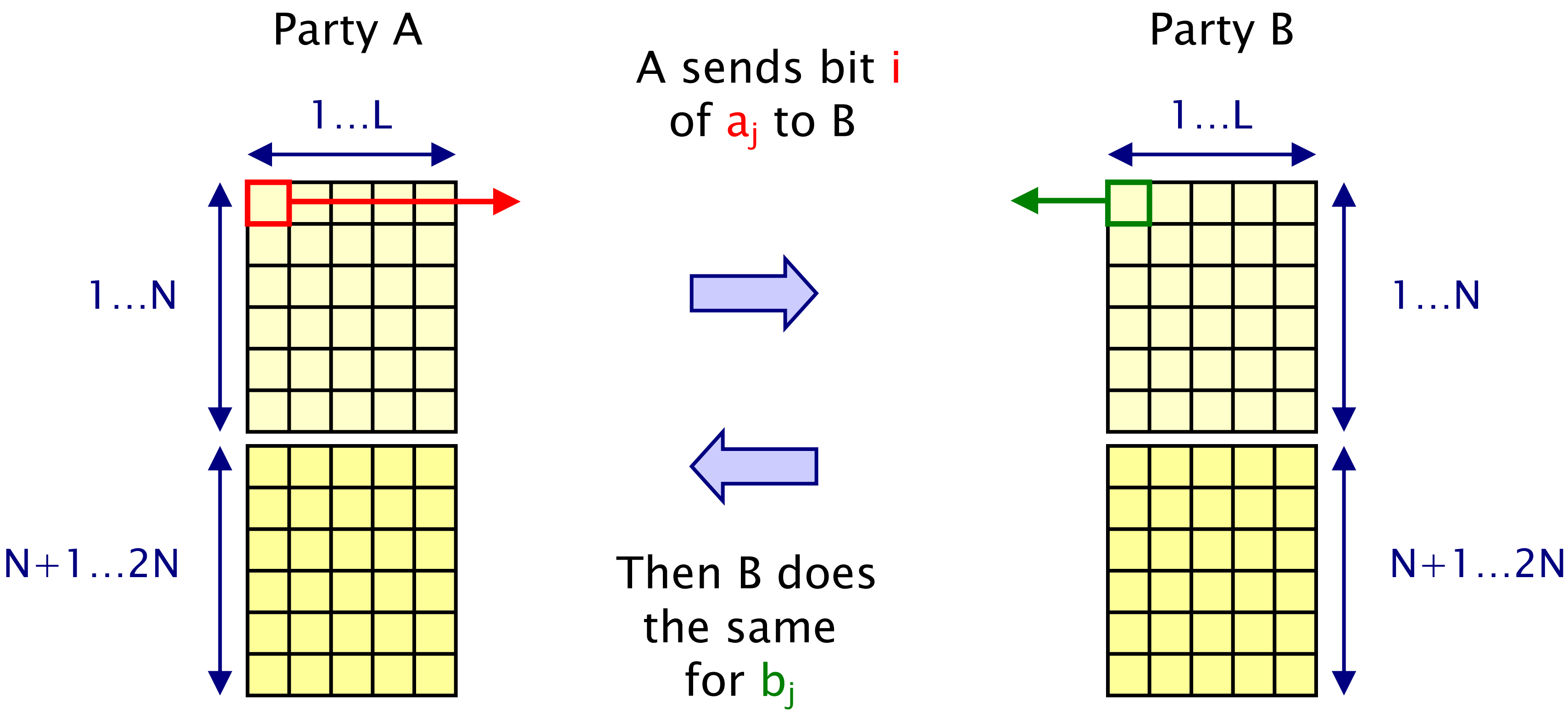


(after $j=1\ldots N$, send $j=N+1\ldots 2N$)
(then repeat over $i=1\ldots L$)

Contract signing – EGL3

(step 1)
…
(step 2)
for ($i=1,\ldots,L$) for ($j=1,\ldots,N$)
    A transmits bit i of secret a_j to B
    B transmits bit i of secret b_j to A
for ($i=1,\ldots,L$) for ($j=N+1,\ldots,2N$)
    A transmits bit i of secret a_j to B
    B transmits bit i of secret b_j to A

Modified step 2 for EGL3

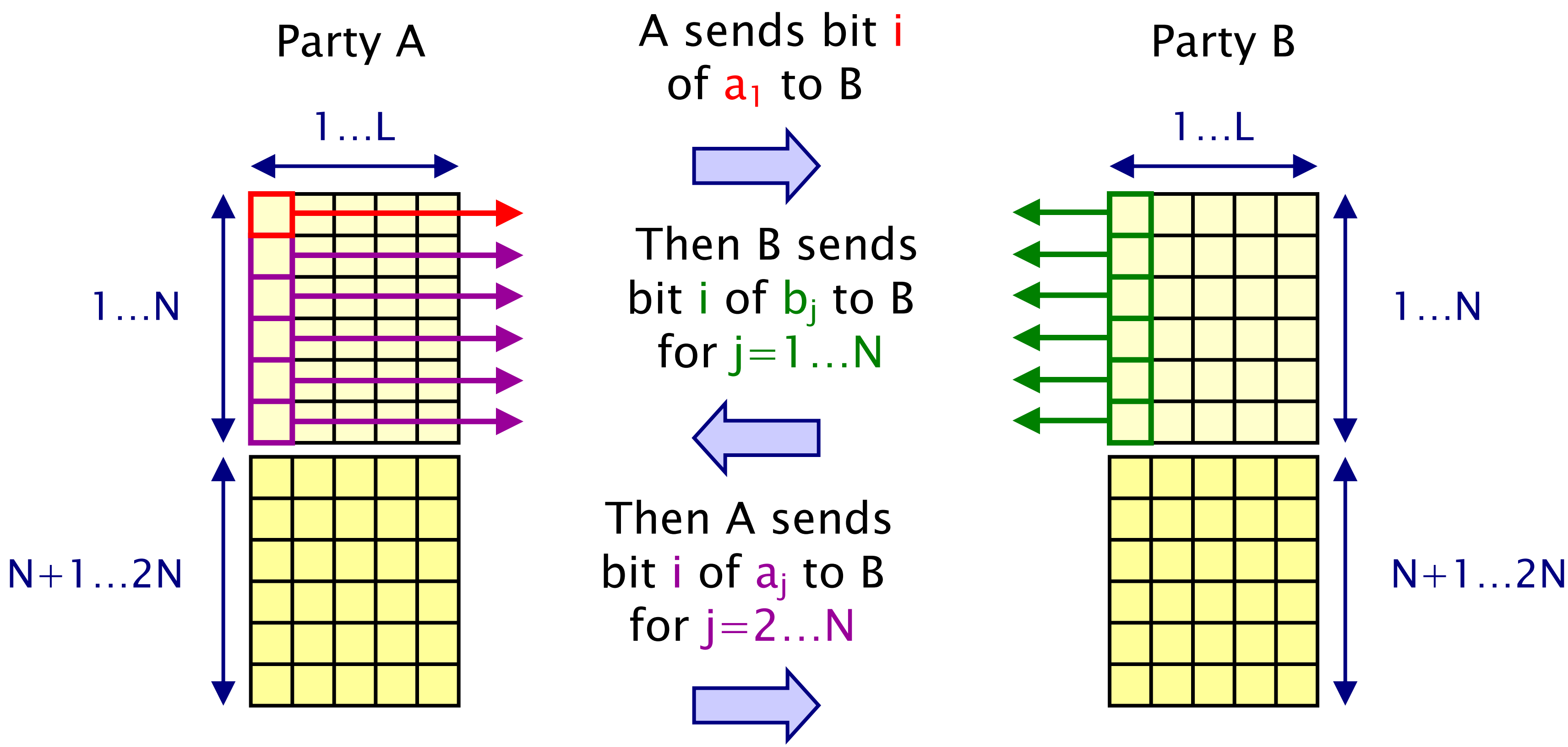


(repeat for j=1…N and for i=1…L)
(then send j=N+1…2N for i=1…L)

Contract signing – EGL4

(step 1)
…
(step 2)
for (i=1,…,L)
    A transmits bit i of secret a_1 to B
    for (j=1,…,N) B transmits bit i of secret b_j to A
    for (j=2,…,N) A transmits bit i of secret a_j to B
for (i=1,…,L)
    A transmits bit i of secret a_{N+1} to B
    for (j=N+1,…,2N) B transmits bit i of secret b_j to A
    for (j=N+2,…,2N) A transmits bit i of secret a_j to B

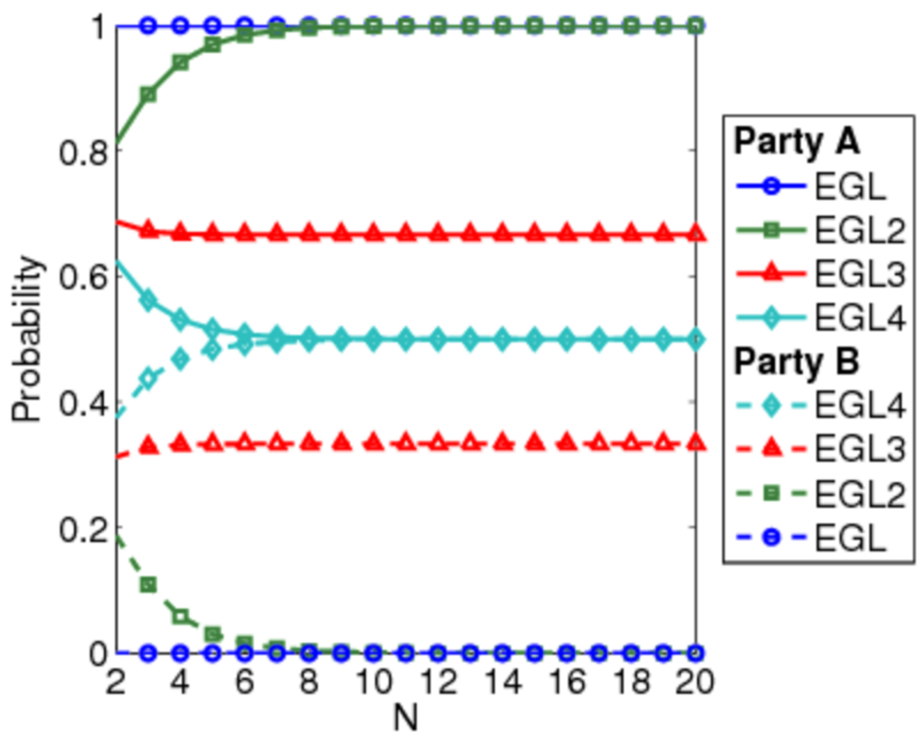
Modified step 2 for EGL4



(repeat for i=1…L)
(then send j=N+1…2N in same fashion)

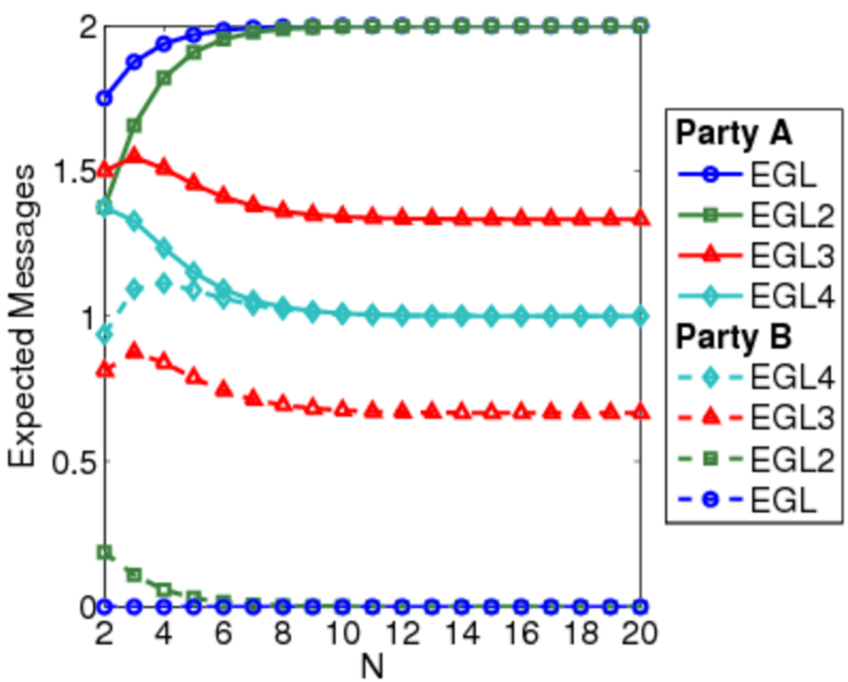
Contract signing – Results

- The chance that the protocol is unfair (N = secrets)
  - probability that one party gains knowledge first
  - $P_{=?}$ [F ($know_B \wedge \neg know_A$)] and $P_{=?}$ [F ($know_A \wedge \neg know_B$)]



Contract signing – Results

- The influence that each party has on the fairness
  - once a party knows a pair, the expected number of messages from this party required before the other party knows a pair



R=? [F $know_A$]

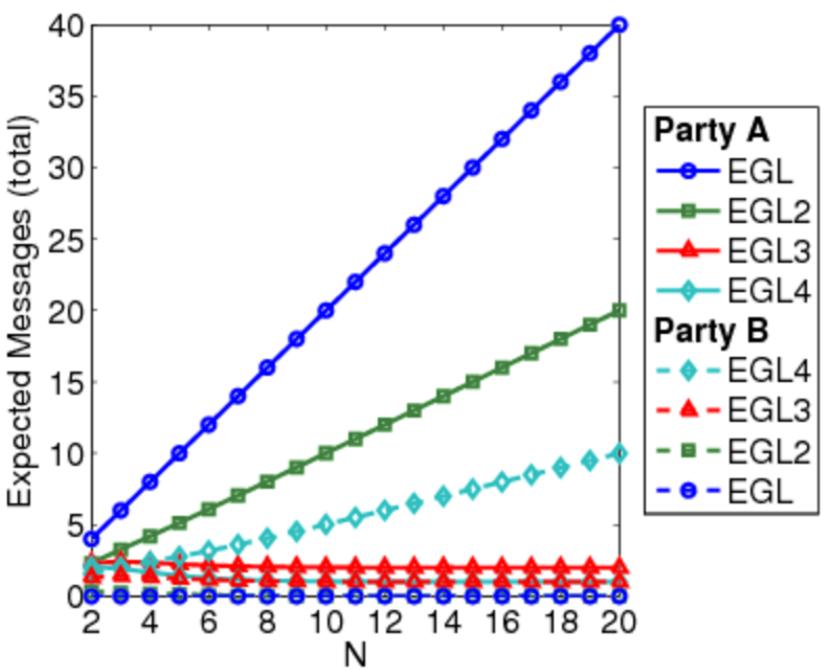
Reward structure:

Assign 1 to transitions corresponding to messages being sent from B to A after B knows a pair

(and 0 to all other transitions)

Contract signing – Results

- The duration of unfairness of the protocol
  - once a party knows a pair, the expected total number of messages that need to be sent before the other knows a pair



R=? [F know_A]

Reward structure:

Assign 1 to transitions corresponding to any message being sent between A and B after B knows a pair

(and 0 to all other transitions)

Contract signing – Results

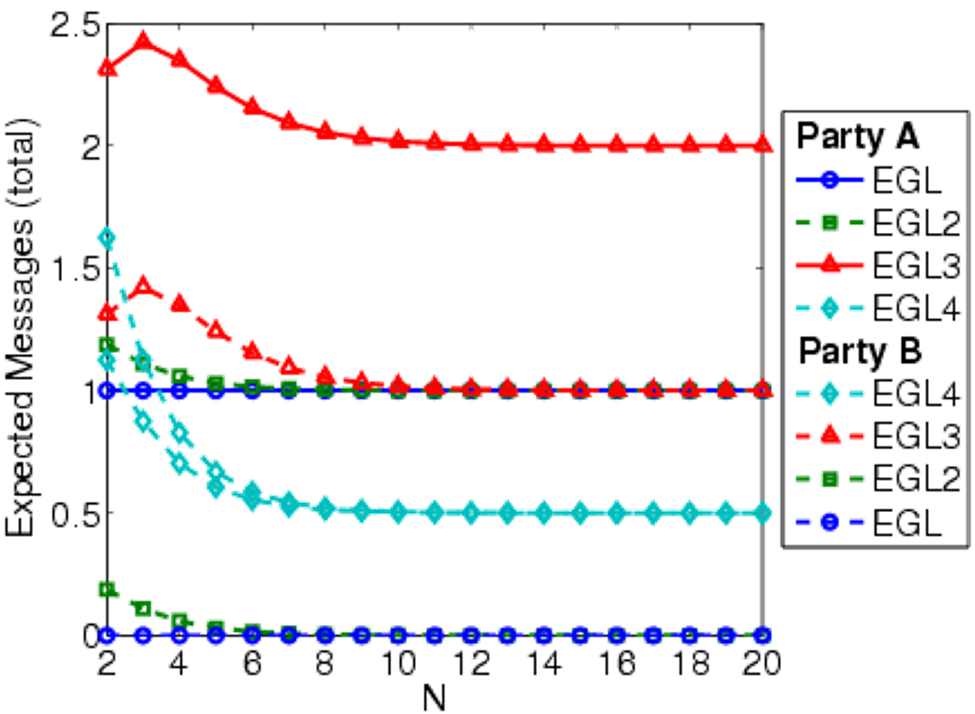
- Results show EGL4 is the 'fairest' protocol

- Except for measure of "duration of unfairness"
  - expected messages that need to be sent for a party to know a pair once the other party knows a pair
  - this value is larger for B than for A
  - and, in fact, as N increases, this measure:
    - increases for B
    - decreases for A

- Solution:
  - if a party sends a sequence of bits in a row (without the other party sending messages in between), require that the party send these bits as a single message

Contract signing – Results

- The duration of unfairness of the protocol
  - (with the solution on the previous slide applied to all variants)



Summing up…

- Costs and rewards
  - real-valued assigned to states/transitions of a DTMC
- Properties
  - expected instantaneous/cumulative reward values
  - PRISM property specifications: adds R operator to PCTL
- Model checking
  - instantaneous: matrix-vector multiplications
  - cumulative: matrix-vector multiplications
  - reachability: graph analysis + linear equation systems
- Case study
  - randomised contract signing