LCD (25/03/2024)

* BISIMILARITY

  * equivalence

  * largest bisimulation

OBSERVATION: Bisimulations need not to be equivalences

$$\begin{tikzcd}
s \arrow[r] \arrow[d, "a"] \arrow[rr, bend left, dashed, "\times"] & t \arrow[r] \arrow[d, "a"] & v \arrow[d, "a"] \\
s_1 \arrow[r] \arrow[d, "b"] & t_1 \arrow[r] \arrow[d, "b"] & v_1 \arrow[d, "b"] \\
s_2 \arrow[r] & t_2 \arrow[r] & v_2
\end{tikzcd}$$

$R = \{ (s,t), (t,v),$
$(s_1,t_1), (t_1,v_1),$
$(s_2,t_2), (t_2,v_2) \}$

bisimulation

not reflexive $\forall P \quad P \not R P$

" symmetric $\forall (P,Q) \quad P R Q \Rightarrow \neg Q R P$

" transitive $\forall P,Q,S \quad P R Q, \; Q R S \Rightarrow \neg P \not R S$

OBSERVATION: given P, Q

$P \sim Q$ iff

$$\boxed{\begin{array}{l} \text{if } P \xrightarrow{\alpha} P' \text{ then } Q \xrightarrow{\alpha} Q' \text{ and } P' \sim Q' \\ \text{if } Q \xrightarrow{\alpha} Q' \text{ then } P \xrightarrow{\alpha} P' \text{ and } P' \sim Q' \end{array}} \quad (*)$$

$(\Rightarrow)$ $\sim$ is a bisimulation

$(\Leftarrow)$ we want to show that assuming $(*)$ we can conclude $P \sim Q$

we construct R bisimulation s.t. $P R Q$

we define

$$R = \{(P,Q)\} \cup \sim$$

R is a bisimulation, i.e. if S,T processes s.t. $S R T$

- if $S \xrightarrow{\alpha} S'$ then $T \xrightarrow{\alpha} T'$ and $S' R T'$
- dual

2 possibilities

① $S = P$, $T = Q$ then by $(*)$ we know that

if $S \xrightarrow{\alpha} S'$ then $T \xrightarrow{\alpha} T'$ and $S' \sim T'$

$\Downarrow$

$S' \mathcal{R} T'$

+ dual

② $S \sim T$

if $S \xrightarrow{\alpha} S'$ since $\sim$ is a bisimulation $T \xrightarrow{\alpha} T'$ and $S' \sim T'$

$\Downarrow$

$S' \mathcal{R} T'$

+ dual

Hence $\mathcal{R} = \{(P,Q)\} \cup \sim$ is a bisimulation $\Rightarrow$ $P \sim Q$ $\square$

We want $P \sim Q$

$\mathcal{R} = \{(P,Q)\} \cup \sim$ bisimulation $\Rightarrow$ $\mathcal{R} \subseteq \sim$

$\Rightarrow$ $(P,Q) \in \sim$

coinductive logics

$$\dfrac{\dfrac{P}{\vdots}}{P}$$

* String bisimilarity

- string bisimulation: $\mathcal{R} \subseteq Proc \times Proc$ s.t. when $P \mathcal{R} Q$

  if $P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} P_n$

  then $Q \xrightarrow{\alpha_1} Q_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} Q_n$ and $P_n \mathcal{R} Q_n$

string bisimilarity $P \sim_{string} Q$ if $P \mathcal{R} Q$ for some $\mathcal{R}$ string bisim.

EXERCISE (exam) : show $\sim \; = \; \sim_{string}$

<u>OBSERVATION</u> : Given P, Q if $P \sim Q$ then $Traces(P) = Traces(Q)$

<u>proof</u>

assume $P \sim Q$ ($\Rightarrow$ $P \sim_{string} Q$)

and let $P \xrightarrow{\alpha_1} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} P_n \in Traces(P)$

then $Q \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} Q_n \in Traces(Q)$

(& $P_n \underset{string}{\sim} Q_n$)

$\Rightarrow$ $Traces(P) \subseteq Traces(Q)$

and then by symmetry since $Q \sim P$ $Traces(Q) \subseteq Traces(P)$

whence equality.

<u>EXERCISE</u> : $CTraces(P) = \{ \alpha_1 \dots \alpha_n \mid P \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} P_n \not\to \}$

$P \sim Q \overset{?}{\text{then}} CTraces(P) = CTraces(Q)$

<u>OBSERVATION</u> : if $K \overset{def}{=} P$ then $K \sim P$

because there is a bisimulation

$$R = \{(K, P)\} \cup I$$

$$\frac{P \xrightarrow{\alpha} P'}{K \xrightarrow{\alpha} P'}$$

<u>OBSERVATION</u> :

(i) $P \mid Q \sim Q \mid P$

(ii) $P \mid (Q \mid R) \sim (P \mid Q) \mid R$

(iii) $P \mid 0 \sim P$

(i) define a bisimulation

$$R = \{ (P'|Q' \,,\, Q'|P') \mid P', Q' \text{ processes} \} \qquad !$$

$$\frac{P' \xrightarrow{\alpha} P''}{P'|Q' \longrightarrow P''|Q'} \qquad\qquad \frac{P' \xrightarrow{\alpha} P''}{Q'|P' \xrightarrow{\alpha} Q'|P''}$$

(ii) $R = \{ (P'|(Q'|R') \,,\, (P'|Q')|R') \mid P', Q', R' \text{ processes} \}$

(iii) $R = \{ (P'|0 \,,\, P') \mid P' \text{ process} \}$

EXERCISE :

(i) $P \mid (Q+S) \;\overset{?}{\cancel{\sim}}\; (P|Q) + (P|S)$

$(\overline{pub}.\,coffee.0) \mid (\overline{coffee}.0 + \overline{tea}.0)$ $\qquad$ $(\overline{pub}.\,coffee.0 \mid \overline{coffee}.0) +$

$(\overline{pub}.\,coffee.0 \mid \overline{tea}.0)$

(ii) $(P|Q)\setminus L \;\not\sim\; (P\setminus L) \mid (Q \setminus L)$

$(coffee.0 \mid \overline{coffee}.0) \setminus coffee$ $\qquad$ $(coffee.0)\setminus coffee \mid (\overline{coffee}.0)\setminus coffee$

$\downarrow \tau$ $\qquad$ $\not\downarrow \tau$

(iii) $(P|Q)[f] \;\overset{?}{\sim}\; P[f] \mid Q[f]$

$(coffee.0 \mid \overline{tea}.0)\,[\overbrace{b/coffee,\; b/tea}^{f}]$ $\qquad$ $coffee.0[f] \mid \overline{tea}.0[f]$

$b \swarrow \quad \searrow \bar{b}$ $\qquad$ $b \swarrow \quad \downarrow \tau \quad \searrow \bar{b}$

* Bisimilarity is compositional (congruence)

We want that if $P \sim Q$ then for all $C[\ ]$ $C[P] \sim C[Q]$

Let P, Q, R processes, with $P \sim Q$

Then

(i) $\alpha . P \sim \alpha . Q$

(ii) $P + R \sim Q + R$

(iii) $P | R \sim Q | R$

(iv) $P \backslash L \sim Q \backslash L$

(v) $P[f] \sim Q[f]$

<u>proof</u>

we want $\alpha . P \sim \alpha . Q$ and we prove it by building a bisimulation

$$\mathcal{R} = \{ (\alpha . P, \alpha . Q) \} \cup \sim$$

in fact for all S, T s.t. $S \mathcal{R} T$

if $S \xrightarrow{\alpha} S'$ then $T \xrightarrow{\alpha} T'$ and $S' \mathcal{R} T'$

+ dual

there are two possibilities for S, T

(1) $S = \alpha . P$, $T = \alpha . Q$

$\alpha . P \xrightarrow{\alpha} P$ $\rightsquigarrow$ $\alpha . Q \xrightarrow{\alpha} Q$ and $P \sim Q$
$\Downarrow$
$P \mathcal{R} Q$

+ dual

(2) $S \sim T$

same as in observation above

(ii) $P + R \sim Q + R$

$\mathcal{R} = \{ \underbrace{(P+R,\ Q+R)} \} \cup \sim$ is a bisimulation

$P + R \xrightarrow{\alpha} \_ \quad \rightsquigarrow \quad Q + R \xrightarrow{\alpha} \_$

$\mathcal{R}$

2 possibilities

(1) $\dfrac{\boxed{P \xrightarrow{\alpha} P'}}{\boxed{P + R \xrightarrow{\alpha} P'}}$ & $\boxed{P \sim Q}$

$\dfrac{Q \xrightarrow{\alpha} Q'}{\boxed{Q + R \xrightarrow{\alpha} Q'}}$ & $\boxed{P' \sim Q'}$

$\downarrow$

$P' \ \mathcal{R} \ Q'$

(2) $\dfrac{R \xrightarrow{\alpha} R'}{\boxed{P + R \xrightarrow{\alpha} R'}}$

then

$\dfrac{R \xrightarrow{\alpha} R'}{\boxed{Q + R \xrightarrow{\alpha} R'}}$ & $R' \ \mathcal{R} \ R'$

ok since $R' \sim R'$ by reflexity

$\Downarrow$

$R' \ \mathcal{R} \ R'$

(iii) $P \mid R \sim Q \mid R$

define $\mathcal{R} = \{ \cancel{(P \mid R,\ Q \mid R)} \} \cup \sim$

$\dfrac{P \xrightarrow{\alpha} P'}{P \mid R \xrightarrow{\alpha} \underline{P' \mid R}}$ since $P \sim Q$ $\dfrac{Q \xrightarrow{\alpha} Q'}{Q \mid R \xrightarrow{\alpha} \underline{Q' \mid R}}$ $P' \sim Q'$

?

$\mathcal{R}$

<u>Idea</u>:

$$R = \{ (P' | R', Q' | R') \mid P' \sim Q' \text{ and } R' \text{ process} \}$$

bisimulation

(IV) $P \setminus L \sim Q \setminus L$

$$R = \{ (P' \setminus L, Q' \setminus L) \mid P' \sim Q' \}$$

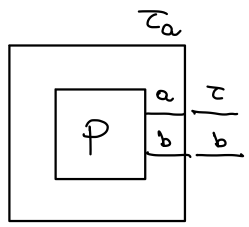
(V) $P[f] \sim Q[f]$

$$R = \{ (P'[f], Q'[f]) \mid P' \sim Q' \}$$ bisimulation

$\square$

<u>EXERCISE</u>

given P process



$L \subseteq \mathcal{A}$

$$\frac{P \xrightarrow{\alpha} P'}{\tau_L(P) \xrightarrow{\alpha} \tau_L(P')} \quad \alpha, \bar{\alpha} \notin L$$

$$\frac{P \xrightarrow{\alpha} P'}{\tau_L(P) \xrightarrow{\tau} \tau_L(P')} \quad \alpha, \bar{\alpha} \in L$$

This can be encoded in the language, i.e. there is a context $C_L[\ ]$ s.t.

$$C_L[P] \sim \tau_L(P)$$

[forbidden τ/a]

EXERCISE: (Bisimulation Up-to)

A bisimulation up to (bisimilarity) is a relation $R \subseteq Proc \times Proc$ such that if $P \, R \, Q$

- → if $P \xrightarrow{\alpha} P'$ then $Q \xrightarrow{\alpha} Q'$ and $P' \sim P'' \, R \, Q'' \sim Q'$
- → dual

$$\boxed{\text{If } R \text{ is a bisimulation up to and } P \, R \, Q \text{ then } P \sim Q}$$

EXERCISE

IDEA: I show if R is ~~bisimulation~~ up to then R bisimulation

NOT TRUE