Lecture 2
Discrete-time Markov Chains

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Probabilistic Model Checking

- Formal verification and analysis of systems that exhibit probabilistic behaviour
  - e.g. randomised algorithms/protocols
  - e.g. systems with failures/unreliability
- Based on the construction and analysis of precise mathematical models
- This lecture: discrete-time Markov chains

Overview

- Probability basics
- Discrete-time Markov chains (DTMCs)
  - definition, properties, examples
- Formalising path-based properties of DTMCs
  - probability space over infinite paths
- Probabilistic reachability
  - definition, computation
- Sources and further reading: Section 10.1 of [BK08]

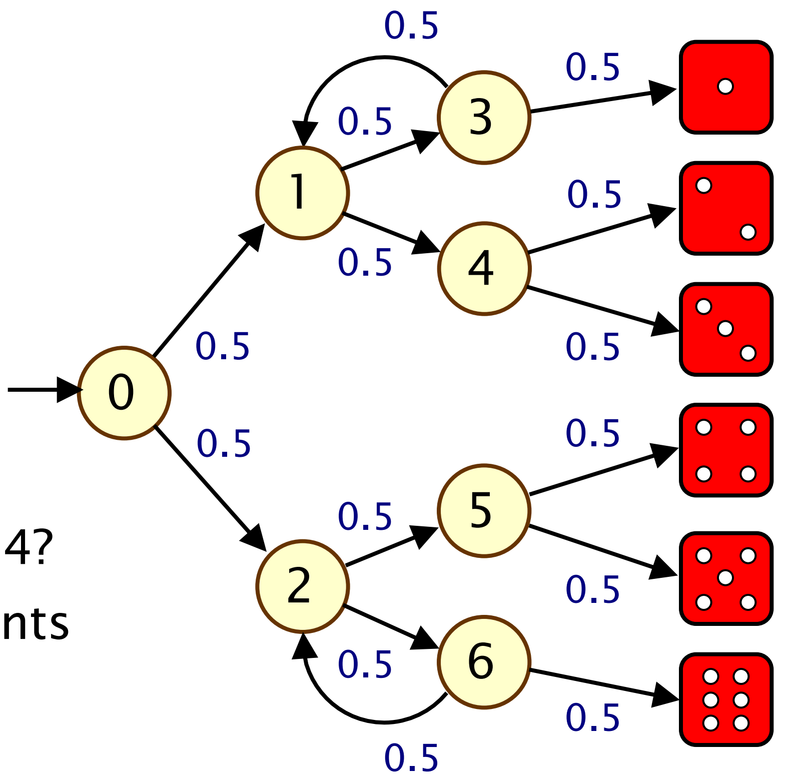
Probability basics

- First, we need an experiment
  - The sample space Ω is the set of possible outcomes
  - An event is a subset of Ω, can form events $A \cap B$, $A \cup B$, $\Omega \setminus A$

- Examples:
  - toss a coin: $\Omega = \{H,T\}$, events: "H", "T"
  - toss two coins: $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$, event: "at least one H"
  - toss a coin ∞-often: Ω is set of infinite sequences of H/T event: "H in the first 3 throws"
- Probability is:
  - Pr("H") = Pr("T") = 1/2, Pr("at least one H") = 3/4
  - Pr("H in the first 3 throws") = 1 - 1/8 = 7/8

Probability example

- Modelling a 6-sided die using a fair coin
  - algorithm due to Knuth/Yao:
  - start at 0, toss a coin
  - upper branch when H
  - lower branch when T
  - repeat until value chosen

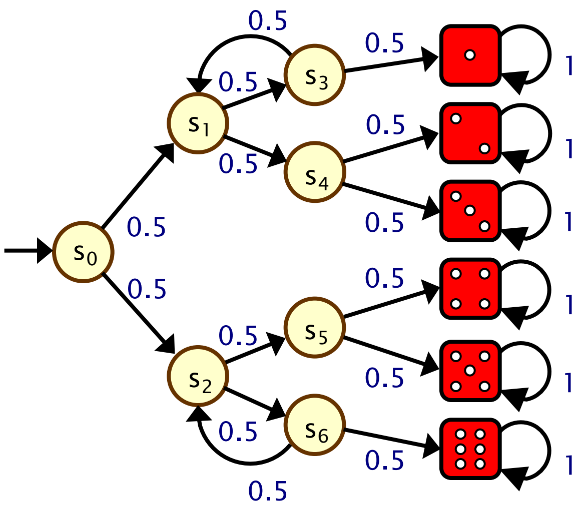


- Is this algorithm correct?
  - e.g. probability of obtaining a 4?
  - obtain as disjoint union of events
  - THH, TTTHH, TTTTTHH, …
  - Pr("eventually 4")

    $= (1/2)^3 + (1/2)^5 + (1/2)^7 + \ldots = 1/6$

Example…

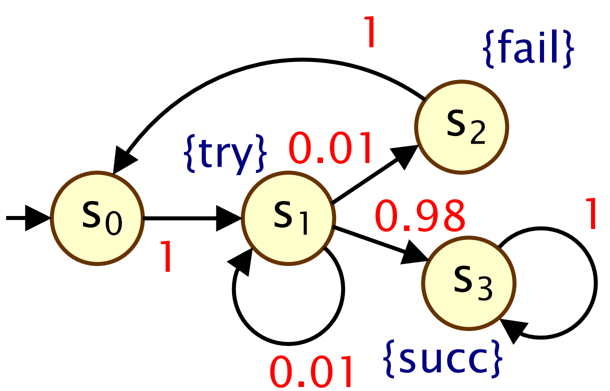
- Other properties?
  - “what is the probability of termination?”
- e.g. efficiency?
  - “what is the probability of needing more than 4 coin tosses?”
  - “on average, how many coin tosses are needed?”



- Probabilistic model checking provides a framework for these kinds of properties: we need to discuss
  - modelling languages
  - property specification languages
  - model checking algorithms, techniques and tools

Discrete-time Markov chains

- State-transition systems augmented with probabilities
- States
  - set of states representing possible configurations of the system being modelled
- Transitions
  - transitions between states model evolution of system's state; occur in discrete time-steps
- Probabilities
  - probabilities of making transitions between states are given by discrete probability distributions
- Labels

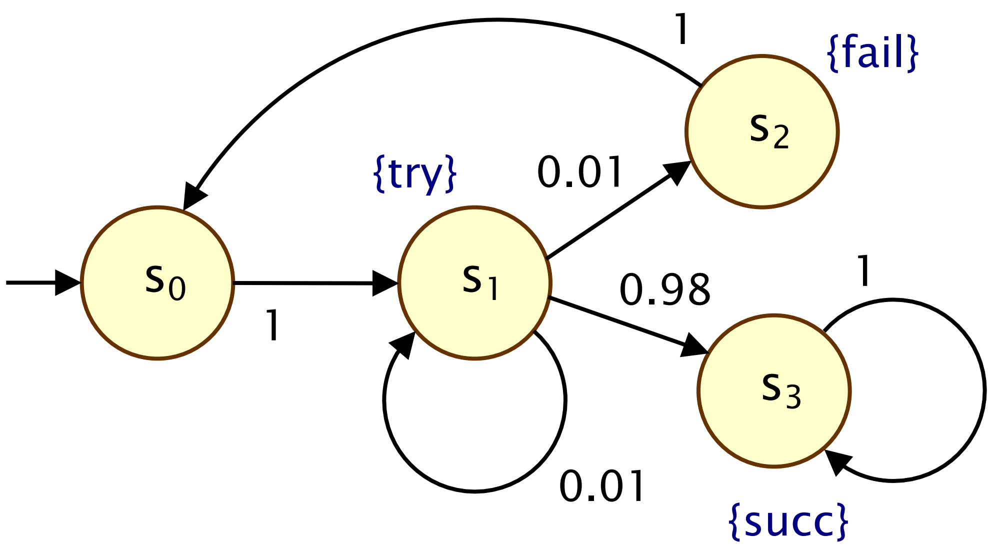


Markov property

- If the current state is known (namely, “conditional on current state”), then future states of the system are independent of its past states
- i.e. the current state of the model contains all information that can influence the future evolution of the system
- also known as “memoryless-ness”

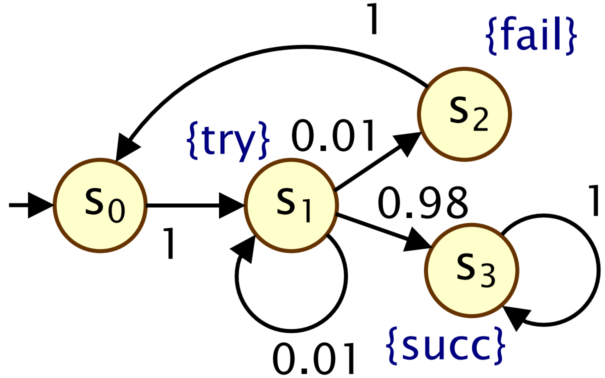
Simple DTMC example

- Modelling a very simple communication protocol
  - after one step, process starts trying to send a message
  - with probability 0.01, channel not ready so wait a step
  - with probability 0.98, send message successfully and stop
  - with probability 0.01, message sending fails, thus restart



Discrete-time Markov chains

- Formally, a DTMC D is a tuple $(S,s_{init},\mathbf{P},L)$ where:
  - S is a set of states (S is known as the "state space")
  - $s_{init} \in S$ is the initial state
  - $\mathbf{P} : S \times S \rightarrow [0,1]$ is the transition probability matrix where $\Sigma_{s' \in S} \mathbf{P}(s,s') = 1$ for all $s \in S$
  - $L : S \rightarrow 2^{AP}$ is function labelling states with atomic propositions (taken from a finite set AP)



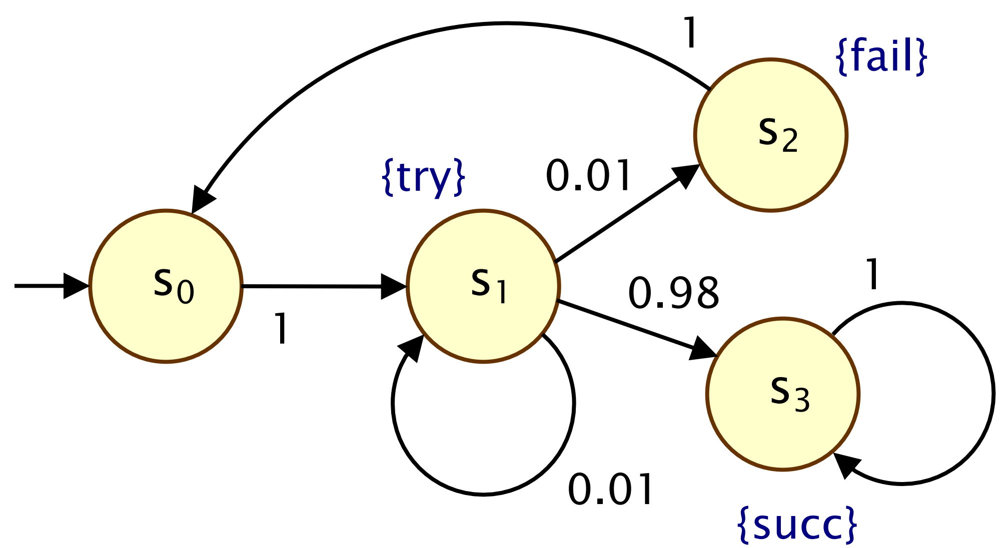
Simple DTMC example

$D = (S, s_{init}, \mathbf{P}, L)$

$S = \{s_0, s_1, s_2, s_3\}$
$s_{init} = s_0$

$AP = \{try, fail, succ\}$
$L(s_0) = \emptyset$,
$L(s_1) = \{try\}$,
$L(s_2) = \{fail\}$,
$L(s_3) = \{succ\}$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Some more terminology

- **P** is a stochastic matrix, meaning it satisifes:
  - **P**(s,s') $\in$ [0,1] for all s,s' $\in$ S and $\Sigma_{s' \in S}$ **P**(s,s') = 1 for all s $\in$ S
- A sub-stochastic matrix satisfies:
  - **P**(s,s') $\in$ [0,1] for all s,s' $\in$ S and $\Sigma_{s' \in S}$ **P**(s,s') $\leq$ 1 for all s $\in$ S
- An absorbing state is a state s for which:
  - **P**(s,s) = 1 and **P**(s,s') = 0 for all s$\neq$s'
  - the transition from s to itself is sometimes called a self-loop
- Note: Since we assume **P** is stochastic…
  - every state has at least one outgoing transition
  - i.e. no deadlocks (in model checking terminology)

DTMCs: An alternative definition

- Alternative definition… a DTMC is:
  - a family of random variables { X(k) | k=0,1,2,… }
  - where X(k) are r.v. values at discrete time steps
  - i.e. X(k) is the state of the system at time step k
  - which satisfies:

- The Markov property ("memoryless-ness")
  - $Pr(X(k)=s_k \mid X(k-1)=s_{k-1}, \ldots , X(0)=s_0)$
    $= Pr(X(k)=s_k \mid X(k-1)=s_{k-1})$
  - for a given current state, future states are independent of past

- This allows us to adopt the "state-based" view presented so far (which is better suited to this context)

Other assumptions made here

- We consider time-homogenous DTMCs
  - transition probabilities are independent of time step k:
  - $\Pr(X(k)=s_k \mid X(k-1)=s_{k-1}) = \mathbf{P}(s_{k-1},s_k)$
  - otherwise: time-inhomogenous (tricky instance)

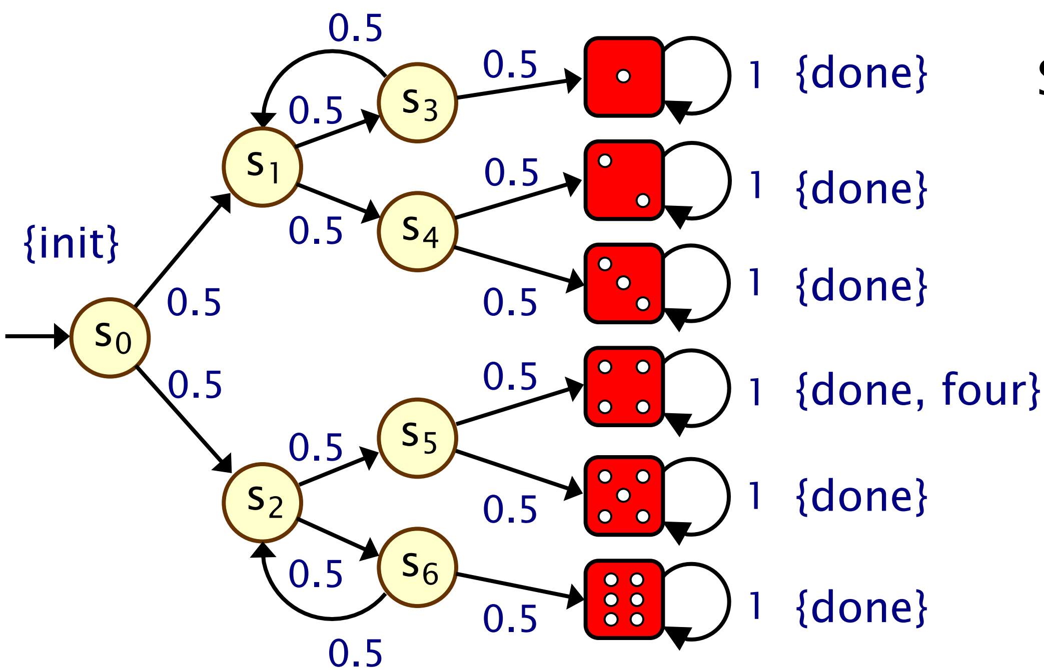
- We will (mostly) assume that the state space S is finite
  - in general, S can be a countable set

- Initial state $s_{init} \in S$ can be generalised…
  - to an initial probability distribution $s_{init} : S \rightarrow [0,1]$

- Transition probabilities are reals: $\mathbf{P}(s,s') \in [0,1]$
  - but for algorithmic purposes, are assumed to be rationals

DTMC example 2 - Coins and dice

- Recall Knuth/Yao's die algorithm from earlier:



$S = \{ s_0, s_1, \ldots, s_6, 1, 2, \ldots, 6 \}$

$s_{init} = s_0$

$\mathbf{P}(s_0,s_1)=0.5$
$\mathbf{P}(s_0,s_2)=0.5$
etc.

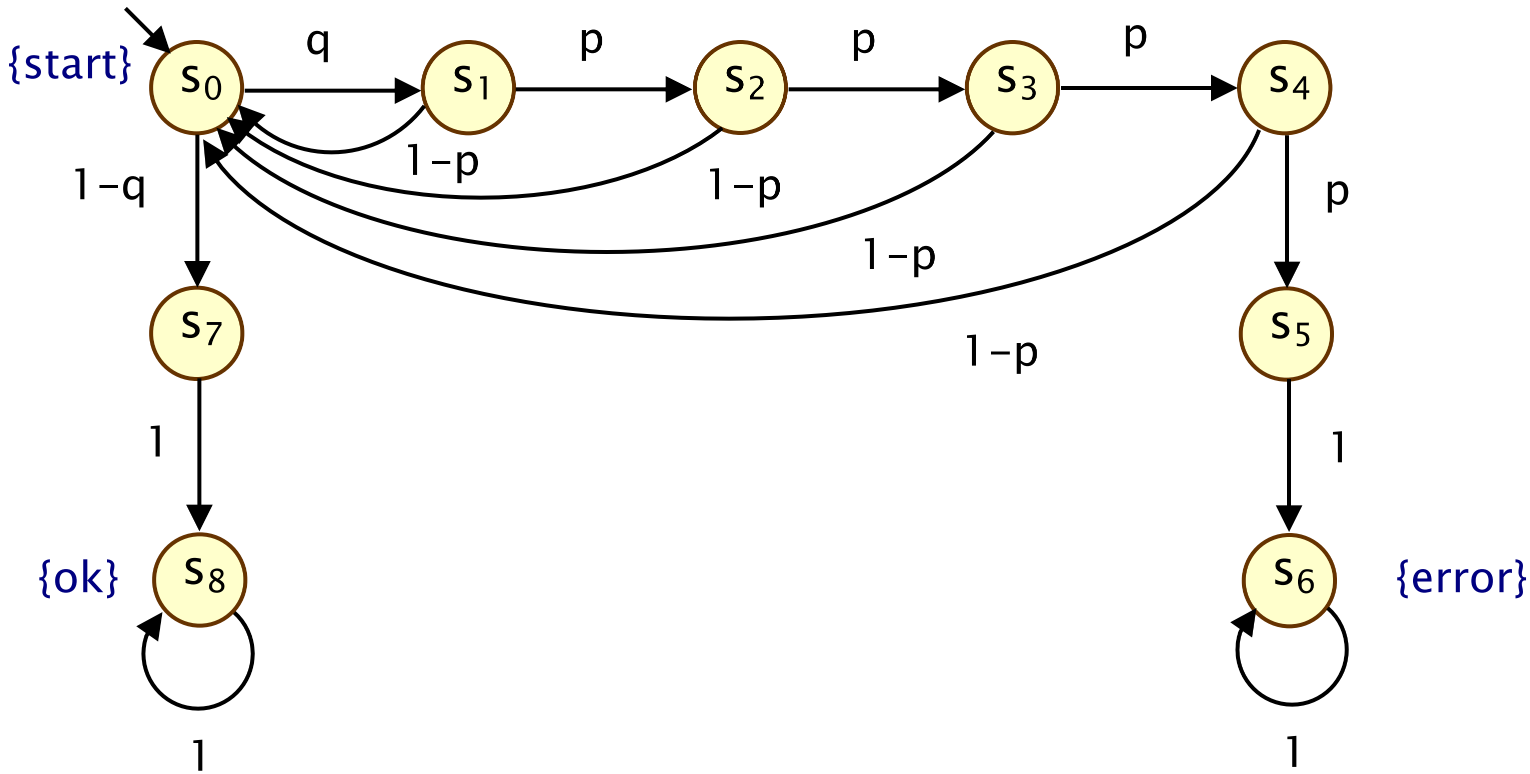
$L(s_0) = \{init\}$
etc.

DTMC example 3 – Zeroconf

- Zeroconf = "Zero configuration networking"
  - self-configuration for local, ad-hoc networks
  - automatic configuration of unique IP for new devices
  - simple; no DHCP, DNS, …
- Basic idea:
  - 65,024 available IP addresses (IANA-specified range)
  - new node picks address U at random
  - broadcasts "probe" messages: "Who is using U?"
  - a node already using U replies to the probe
  - in this case, protocol is restarted
  - messages may not get sent (transmission fails, host busy, …)
  - so: nodes send multiple (n) probes, waiting after each one

DTMC for Zeroconf

- $n=4$ probes, m existing nodes in network
- probability of message loss: p
- probability that new address is in use: q = m/65024

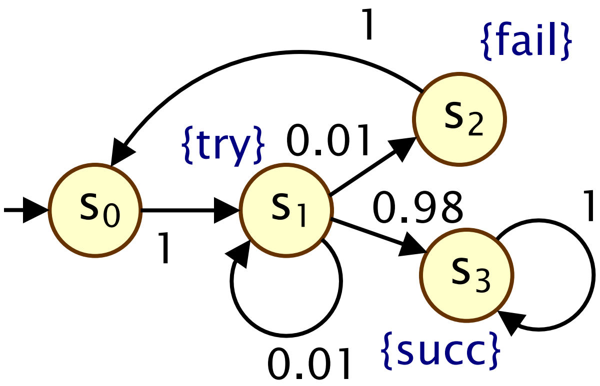


Properties of DTMCs

- Path-based properties
  - what is the probability of observing a particular behaviour (or class of behaviours)?
  - e.g. "what is the probability of throwing a 4?"
- Transient properties
  - probability of being in state s after t steps?
- Steady state
  - long-run probability of being in each state
- Expectations
  - e.g. "what is the average number of coin tosses required?"

DTMCs and paths

- A path in a DTMC represents an execution (i.e. one possible behaviour) of the system being modelled
- Formally:
  - infinite sequence of states $s_0 s_1 s_2 \ldots$ such that $\mathbf{P}(s_i, s_{i+1}) > 0$, $\forall i \geq 0$
  - infinite unfolding of DTMC (no blocking conditions)

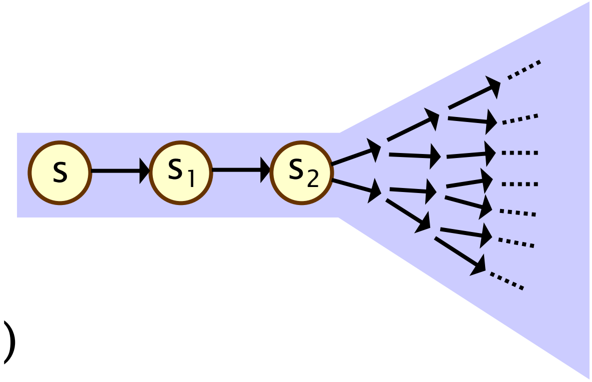


- Examples:
  - never succeeds: $(s_0 s_1 s_2)^\omega$
  - tries, waits, fails, retries, succeeds: $s_0 s_1 s_1 s_2 s_0 s_1 (s_3)^\omega$
- Notation:
  - Path(s) = set of all infinite paths starting in state s
  - can also define finite-length paths:
  - $\text{Path}_{\text{fin}}(s)$ = set of all finite paths starting in state s

Paths and probabilities

- To reason (quantitatively) about this system
  - need to define a probability space over paths

- Intuitively:
  - sample space: Path(s) = set of all infinite paths from a state s
  - events: sets of infinite paths from s
  - basic events: cylinder sets (or "cones")
  - cylinder set Cyl(ω), for a finite path ω = set of infinite paths with the common finite prefix ω
  - for example: Cyl(ss_1s_2)



Probability spaces

- Let Ω be an arbitrary non-empty sample set
- A σ-algebra (or σ-field) on Ω is a family Σ of subsets of Ω closed under complementation and countable union, i.e.:
  - if $A \in \Sigma$, the complement $\Omega \setminus A$ is in Σ
  - if $A_i \in \Sigma$ for $i \in \mathbb{N}$, the union $\cup_i A_i$ is in Σ
  - the empty set $\varnothing$ is in Σ
- Elements of Σ are called measurable sets or events
- Theorem: For any family F of subsets of Ω, there exists a unique smallest σ-algebra on Ω containing F

Probability spaces

- Probability space (Ω, Σ, Pr)
  - Ω is the sample space
  - Σ is the set of events: σ-algebra on Ω
  - $Pr : \Sigma \rightarrow [0,1]$ is the probability measure:
    $Pr(\Omega) = 1$ and $Pr(\cup_i A_i) = \Sigma_i Pr(A_i)$ for countable disjoint A_i

Probability space – Simple example

- Sample space Ω
  - $\Omega = \{1,2,3\}$

- Event set Σ
  - e.g. powerset of Ω
  - $\Sigma = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$
  - (closed under complement/countable union, contains $\emptyset$)

- Probability measure Pr
  - e.g. $Pr(1) = Pr(2) = Pr(3) = 1/3$
  - $Pr(\{1,2\}) = 1/3+1/3 = 2/3$, etc.

Probability space – Simple example 2

- Sample space Ω
  - $\Omega = \mathbb{N} = \{ 0,1,2,3,4,\ldots \}$
- Event set Σ
  - e.g. $\Sigma = \{ \emptyset$, "odd", "even", $\mathbb{N} \}$
  - (closed under complement/countable union, contains ∅)
- Probability measure Pr
  - e.g. Pr("odd") = 0.5, Pr("even") = 0.5

Probability space over paths

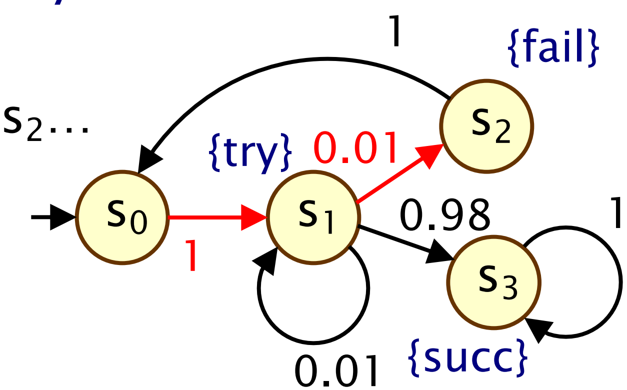
- Sample space Ω = Path(s)
  - set of infinite paths with initial state s
- Event set $\Sigma_{Path(s)}$
  - the cylinder set Cyl(ω) = { $\omega' \in$ Path(s) | ω is prefix of ω' }
  - $\Sigma_{Path(s)}$ is the least σ-algebra on Path(s) containing Cyl(ω) for all finite paths ω starting in s
- Probability measure Pr_s
  - define probability $\mathbf{P}_s(\omega)$ for finite path $\omega = ss_1 \ldots s_n$ as:
    - $\mathbf{P}_s(\omega) = 1$ if ω has length one (i.e. $\omega = s$)
    - $\mathbf{P}_s(\omega) = \mathbf{P}(s,s_1) \cdot \ldots \cdot \mathbf{P}(s_{n-1},s_n)$ otherwise
    - define $Pr_s(Cyl(\omega)) = \mathbf{P}_s(\omega)$ for all finite paths ω
  - Pr_s extends uniquely to a probability measure $Pr_s : \Sigma_{Path(s)} \rightarrow [0,1]$

- See [KSK76] for further details

Paths and probabilities – Example

- Paths where sending fails immediately
  - $\omega = s_0 s_1 s_2$
  - $Cyl(\omega)$ = all paths starting with $s_0 s_1 s_2 \ldots$
  - $P_{s0}(\omega) = P(s_0,s_1) \cdot P(s_1,s_2)$

    $= 1 \cdot 0.01 = 0.01$
  - $Pr_{s0}(Cyl(\omega)) = P_{s0}(\omega) = 0.01$



- Paths which are eventually successful and with no failures
  - $Cyl(s_0s_1s_3) \cup Cyl(s_0s_1s_1s_3) \cup Cyl(s_0s_1s_1s_1s_3) \cup \ldots$
  - $Pr_{s0}(Cyl(s_0s_1s_3) \cup Cyl(s_0s_1s_1s_3) \cup Cyl(s_0s_1s_1s_1s_3) \cup \ldots)$

    $= P_{s0}(s_0s_1s_3) + P_{s0}(s_0s_1s_1s_3) + P_{s0}(s_0s_1s_1s_1s_3) + \ldots$

    $= 1{\cdot}0.98 + 1{\cdot}0.01{\cdot}0.98 + 1{\cdot}0.01{\cdot}0.01{\cdot}0.98 + \ldots$

    $= 0.9898989898\ldots$

    $= 98/99$

Reachability

- Key property: probabilistic reachability
  - probability of a path reaching a state in some target set $T \subseteq S$
  - e.g. "probability of the algorithm terminating successfully?"
  - e.g. "probability that an error occurs during execution?"

- Dual of reachability: invariance
  - probability of remaining within some class of states
  - Pr("remain in set of states T") = 1 - Pr("reach set $S \setminus T$")
  - e.g. "probability that an error never occurs"

- We will also consider other variants of reachability
  - time-bounded, constrained ("until"), …

Reachability probabilities

- Formally: ProbReach(s, T) = Pr_s(Reach(s, T))
  - where Reach(s, T) = { $s_0 s_1 s_2 \ldots \in$ Path(s) | s_i in T for some i }

- Is Reach(s, T) measurable for any $T \subseteq S$? Yes…
  - Reach(s, T) is the union of all basic cylinders Cyl($s_0 s_1 \ldots s_n$) where $s_0 s_1 \ldots s_n$ in $Reach_{fin}$(s, T)
  - $Reach_{fin}$(s, T) contains all finite paths $s_0 s_1 \ldots s_n$ such that: $s_0 = s$, $s_0, \ldots, s_{n-1} \notin T$, $s_n \in T$ (reaches T first time)
  - set of such finite paths $s_0 s_1 \ldots s_n$ is countable

- Probability
  - in fact, the above is a disjoint union
  - so probability obtained by simply summing…

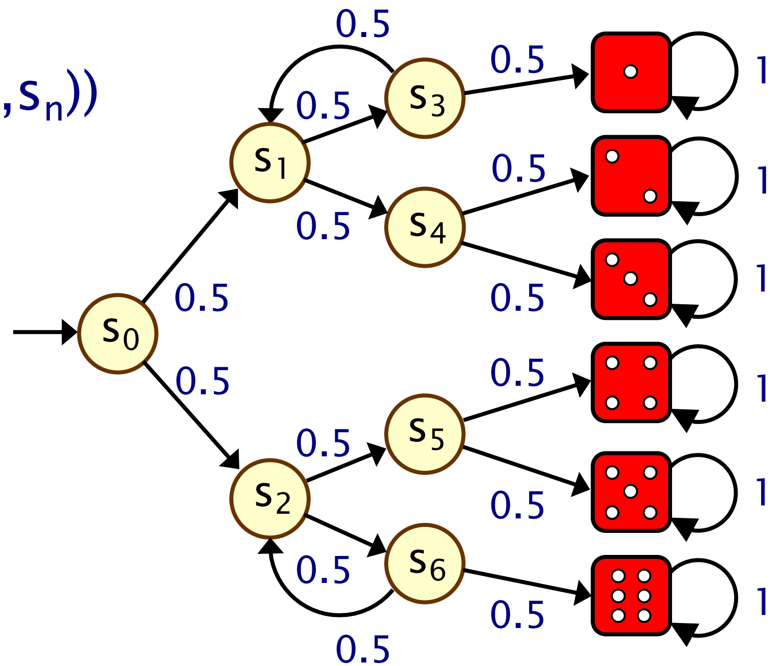
Computing reachability probabilities

- Compute as (infinite) sum…

- $\Sigma_{s_0,\ldots,s_n \in \text{Reachfin}(s,\ T)} \Pr_{s0}(\text{Cyl}(s_0,\ldots,s_n))$

  $= \Sigma_{s_0,\ldots,s_n \in \text{Reachfin}(s,\ T)} \mathbf{P}(s_0,\ldots,s_n)$

- Example:
  - ProbReach(s_0, {4})

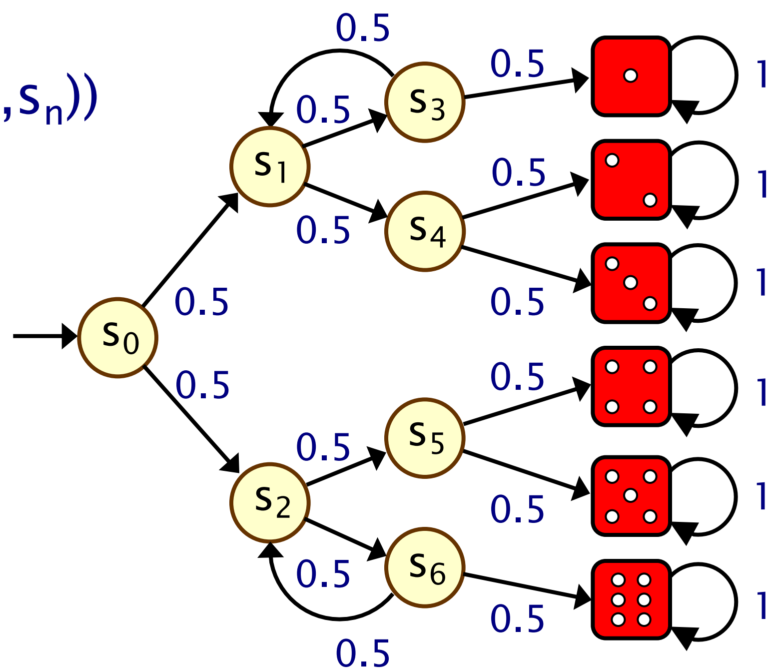


Computing reachability probabilities

- Compute as (infinite) sum…

- $\Sigma_{s_0,\ldots,s_n \in \text{Reachfin}(s,\,T)} \Pr_{s0}(\text{Cyl}(s_0,\ldots,s_n))$

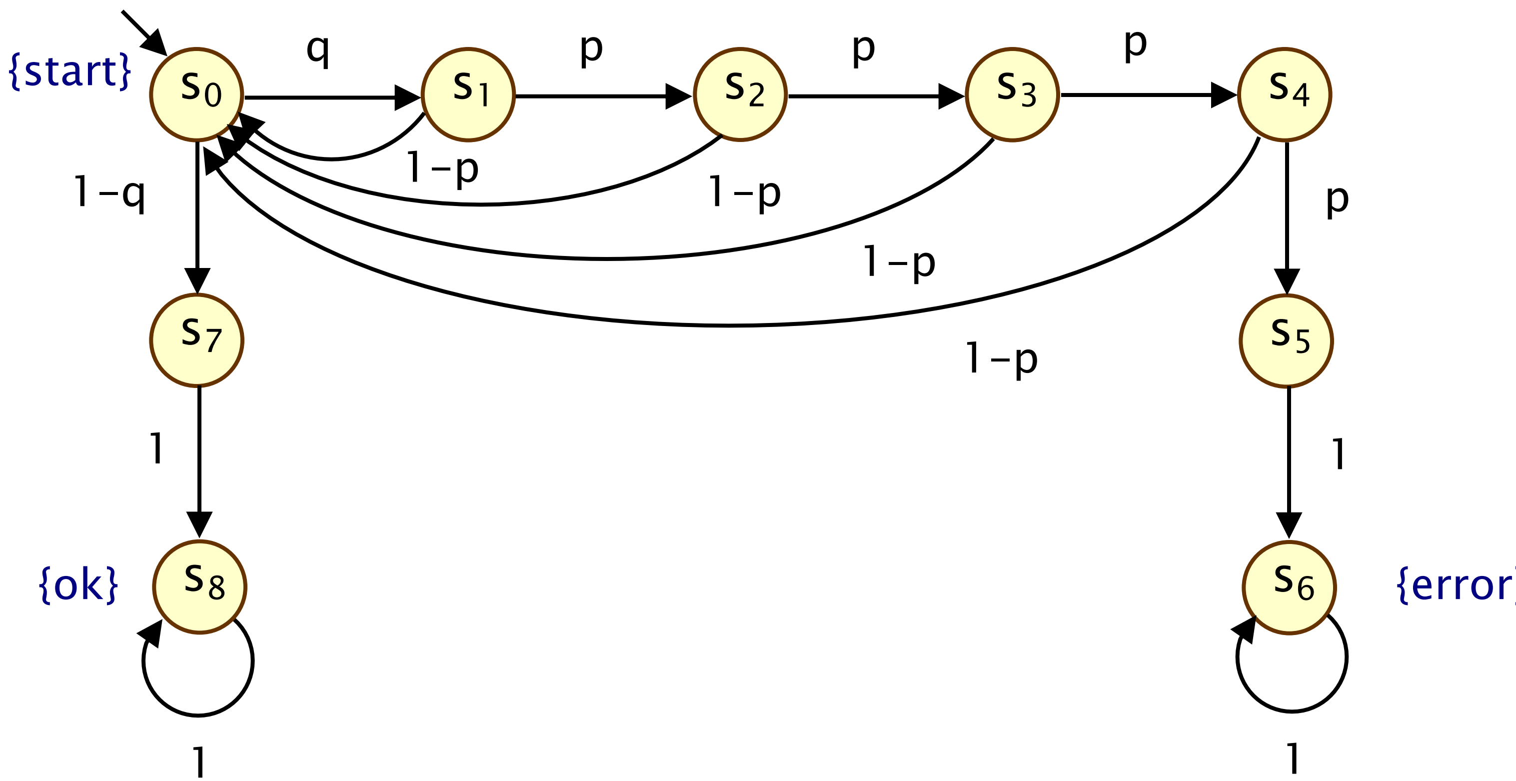
  $= \Sigma_{s_0,\ldots,s_n \in \text{Reachfin}(s,\,T)} \mathbf{P}(s_0,\ldots,s_n)$



- Example:
  - ProbReach(s_0, {4})
  - $= \Pr_{s0}(\text{Reach}(s_0, \{4\}))$
  - Finite path fragments:
  - $s_0(s_2s_6)^n s_2s_5 4$ for $n \geq 0$
  - $\mathbf{P}_{s0}(s_0s_2s_5 4) + \mathbf{P}_{s0}(s_0s_2s_6s_2s_5 4) + \mathbf{P}_{s0}(s_0s_2s_6s_2s_6s_2s_5 4) + \ldots$
  - $= (1/2)^3 + (1/2)^5 + (1/2)^7 + \ldots = 1/6$

Computing reachability probabilities

- ProbReach(s_0, {s_6}) : let us compute as infinite sum …
  - However, this doesn't scale…



Computing reachability probabilities

- Alternative: derive a linear equation system
  - solve for all states simultaneously
  - i.e. compute vector ProbReach(T)

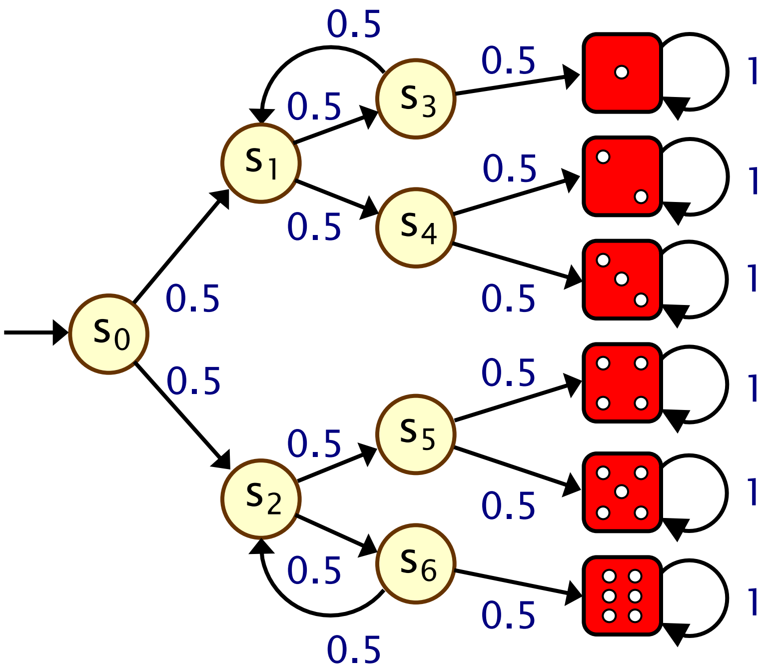
- Let x_s denote ProbReach(s, T)

- Solve:

$$x_s = \begin{cases} 1 & \text{if } s \in T \\ 0 & \text{if } T \text{ is not reachable from } s \\ \sum_{s' \in S} \mathbf{P}(s,s') \cdot x_{s'} & \text{otherwise} \end{cases}$$

Exercise

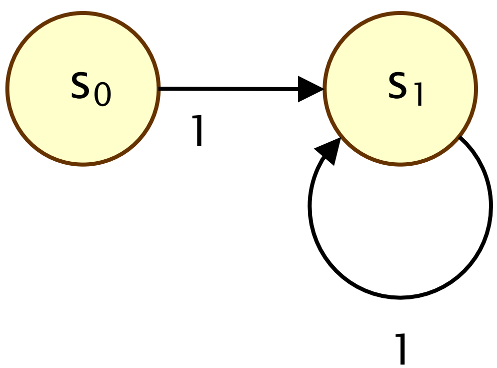
- Compute ProbReach(s_0, {4})



Unique solutions

- Why the need to identify states that cannot reach T?

- Consider this simple DTMC:
  - compute probability of reaching $\{s_0\}$ from s_1



  - linear equation system: $x_{s_0} = 1$, $x_{s_1} = x_{s_1}$
  - multiple solutions: $(x_{s_0}, x_{s_1}) = (1,p)$ for any $p \in [0,1]$

Computing reachability probabilities

- Another alternative: least fixed point characterisation

- Consider functions of the form:
  - $F : [0,1]^{|S|} \rightarrow [0,1]^{|S|}$

vectors of probabilities for each state

- And define:
  - $\underline{y} \leq \underline{y}'$ iff $\underline{y}(s) \leq \underline{y}'(s)$ for all s

- $\underline{y}$ is a fixed point of F if $F(\underline{y}) = \underline{y}$

- A fixed point $\underline{x}$ of F is the least fixed point of F if $\underline{x} \leq \underline{y}$ for any other fixed point $\underline{y}$

Least fixed point

- $\underline{\text{ProbReach}}(T)$ is the least fixed point of the function F:

$$F(\underline{y})(s) = \begin{cases} 1 & \text{if } s \in T \\ \sum_{s' \in S} P(s,s') \cdot \underline{y}(s') & \text{otherwise.} \end{cases}$$

- This yields a simple iterative algorithm to approximate $\underline{\text{ProbReach}}(T)$:

  - $\underline{x}^{(0)} = \underline{0}$ (i.e. $\underline{x}^{(0)}(s) = 0$ for all s)
  - $\underline{x}^{(n+1)} = F(\underline{x}^{(n)})$

  - $\underline{x}^{(0)} \leq \underline{x}^{(1)} \leq \underline{x}^{(2)} \leq \underline{x}^{(3)} \leq \ldots$
  - $\underline{\text{ProbReach}}(T) = \lim_{n \to \infty} \underline{x}^{(n)}$

in practice, terminate when for example:

$$\max_s \left| \underline{x}^{(n+1)}(s) - \underline{x}^{(n)}(s)) \right| < \varepsilon$$

for some user-defined tolerance value ε

Least fixed point

- Expressing ProbReach as a least fixed point…
  - corresponds to solving the linear equation system using the power method
    - other iterative methods exist (see later)
    - power method is guaranteed to converge
  - generalises non-probabilistic reachability
  - can be generalised to:
    - constrained reachability (see PCTL "until")
    - reachability for Markov decision processes
  - also yields step-bounded reachability probabilities…

Bounded reachability probabilities

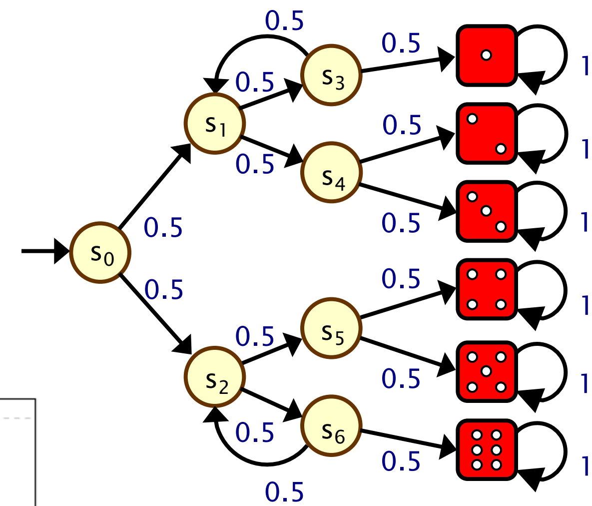
- Probability of reaching T from s within k steps
- Formally: $\text{ProbReach}^{\leq k}(s, T) = \Pr_s(\text{Reach}^{\leq k}(s, T))$ where:
  - $\text{Reach}^{\leq k}(s, T) = \{ s_0 s_1 s_2 \ldots \in \text{Path}(s) \mid s_i \text{ in } T \text{ for some } i \leq k \}$
- $\underline{\text{ProbReach}}^{\leq k}(T) = \underline{x}^{(k+1)}$ from the previous fixed point
  - which gives us…

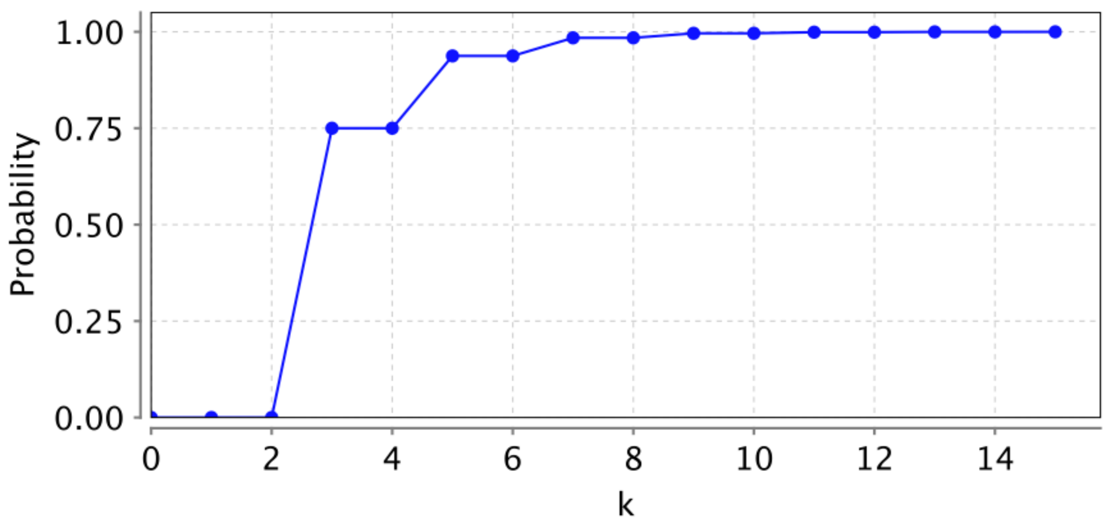
$$\text{ProbReach}^{\leq k}(s, T) = \begin{cases} 1 & \text{if } s \in T \\ 0 & \text{if } k = 0 \,\&\, s \notin T \\ \sum_{s' \in S} \mathbf{P}(s,s') \cdot \text{ProbReach}^{\leq k-1}(s', T) & \text{if } k > 0 \,\&\, s \notin T \end{cases}$$

(Bounded) reachability

- $\text{ProbReach}(s_0, \{1,2,3,4,5,6\}) = 1$

- $\text{ProbReach}^{\leq k}(s_0, \{1,2,3,4,5,6\}) = \ldots$





Summing up…

- Discrete-time Markov chains (DTMCs)
  - state-transition systems augmented with probabilities
- Formalising path-based properties of DTMCs
  - probability space over infinite paths
- Probabilistic reachability
  - infinite sum
  - linear equation system
  - least fixed point characterisation
  - bounded reachability