LCD (18/03/2024)

* BEHAVIOURAL EQUIVALENCE

$$CS = \overline{pub}.\overline{coin}.coffee.CS$$

$$CM = coin.\overline{coffee}.CM$$

$$Office = (CS \mid CM) \setminus \{coin, coffee\}$$

$$Spec = \overline{pub}.Spec$$

$$Office \sim Spec$$

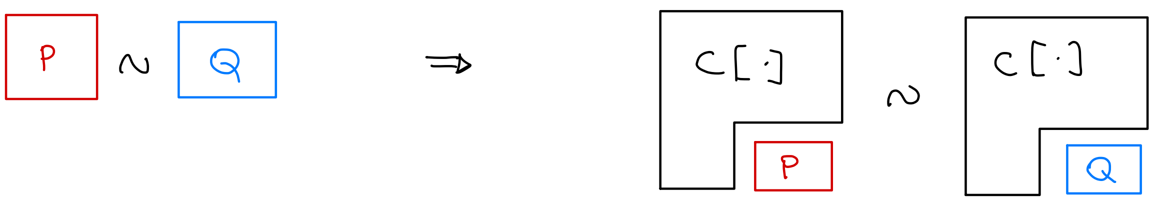
Which properties are expected for $\sim$?

$$\left.\begin{array}{lll} \bullet \text{ reflexive} & P \sim P & \\ \bullet \text{ symmetric} & P \sim Q \Rightarrow Q \sim P & \\ \bullet \text{ transitive} & P \sim Q \text{ and } Q \sim R \text{ then } P \sim R & \end{array}\right\} \text{EQUIVALENCE}$$

$$\underbrace{Spec \sim Sys_1 \sim Sys_2 \sim \dots \sim Sys_n}_{\sim}$$

* congruence / compositionality

if $P \sim Q$ then for every context $C[\ ]$ $C[P] \sim C[Q]$



$$Spec = Spec_1 \mid Spec_2$$

$$\begin{array}{l} Sys_1 \sim Spec_1 \\ Sys_2 \sim Spec_2 \end{array} \Longrightarrow \begin{array}{l} Spec = Spec_1 \mid Spec_2 \\ \qquad \sim Sys_1 \mid Sys_2 \end{array}$$

$$\underset{\substack{\wr \\ Sys_1}}{Spec_1} \mid \underbrace{Spec_2} \sim \underbrace{Sys_1 \mid} \underset{\substack{\wr \\ Sys_2}}{Spec_2} \sim Sys_1 \mid Sys_2$$

Referencial transparency

$P = \text{-- -- --} \underbrace{exp}_{} \text{-- -- --}$

↑ replacing exp. by its value does not alter the behaviour of P

$$\left.\begin{array}{l} x + y = 2 \\ x - y = 1 \end{array}\right\} \;\rightarrow\; y = x - 1$$

$x + x - 1 = 2 \quad \ldots \quad x = \frac{3}{2}$

* <u>the equivalence only depends on the observable behaviour.</u>

OBSERVATIONAL EQUIVALENCE

what is observable ?

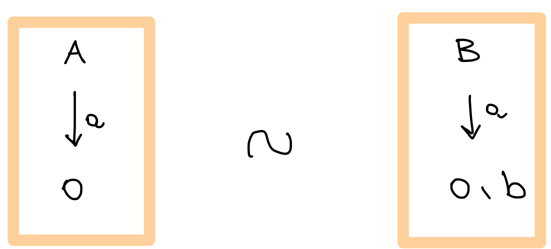
messages
time
cost
⋮

(Van Glabbeek
Linear-time / branching time spectrum)

(0) <u>same transitions</u>

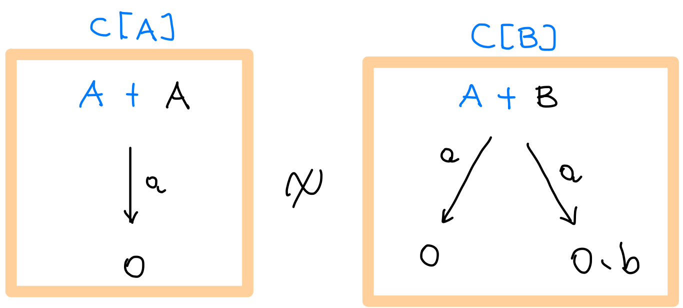
$A = a.0$

$B = (a.0) \backslash b$



$A \sim B$

$C[\;] = A + [\cdot]$



(1) Trace equivalence

$$Tr(P) = \{ \alpha_1 \dots \alpha_n \mid P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} P_2 \cdots \xrightarrow{\alpha_n} P_n \}$$

and define

$$P \sim_T Q \quad \text{iff} \quad Tr(P) = Tr(Q)$$

* equivalence? yes
* observable behaviour only? yes
* compositional? yes

↑ EXAM EXERCISE

Example

$$CTM = coin.(\overline{coffee}.CTM + \overline{tea}.CTM)$$

$$CTM' = coin.\overline{coffee}.CTM' + coin.\overline{tea}.CTM$$

$$Tr(CTM) = Tr(CTM') = (coin\,(\overline{coffee} + \overline{tea}))^* \, coin$$

$$CTM \sim_T CTM'$$

CTM

CTM'

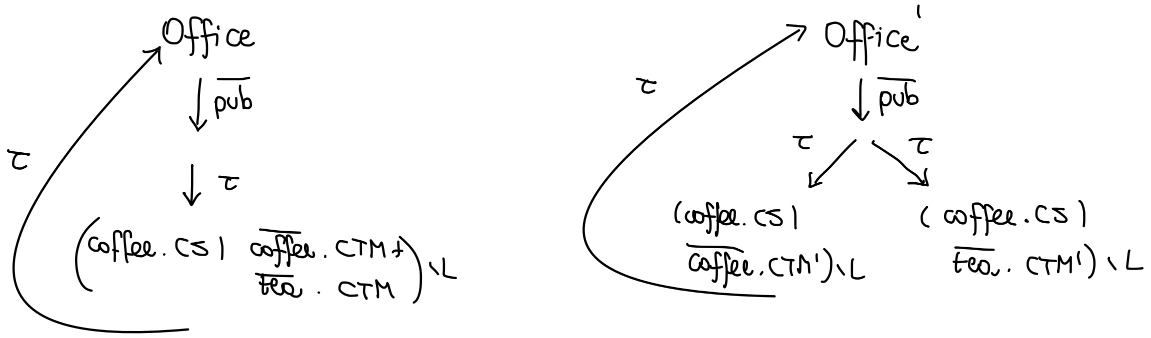
$$CS = \overline{pub}.\overline{coin}.coffee.CS$$

$$Office = (CS \mid CTM) \setminus \{coin, coffee, tea\}$$

$$Office' = (CS \mid CTM') \setminus \{coin, coffee, tea\}$$

$$Office \sim_T Office'$$

i.e. $Tr(Office) = Tr(Office') = (\overline{pub}.\tau.\tau)^* \, (\varepsilon + \overline{pub} + \overline{pub}\,\tau)$



(2) Completed Traces

$$CTr(P) = \{ \alpha_1 \ldots \alpha_n \mid P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} P_2 \ldots \xrightarrow{\alpha_n} P_n \not\to \}$$

and define

$$P \sim_{CT} Q \quad \text{if} \quad P \sim_T Q \quad \text{and} \quad CTr(P) = CTr(Q)$$

Note

$$\underset{\displaystyle \emptyset}{\overset{\Vert}{CTr(Office)}} \neq \underset{\displaystyle \emptyset}{\overset{\#}{CTr(Office')}}$$

$$Office \not\sim_{CT} Office'$$

PROBLEM :

$$CTM \sim_{CT} CTM' \qquad \text{since}$$

$$CTr(CTM) = CTr(CTM') = \emptyset$$

$$Office = (\ CS \mid CTM \ ) \setminus \{coin, coffee, tea\}$$
$$Office' = (\ CS \mid CTM' \ ) \setminus \{coin, coffee, tea\} \qquad \not\sim_{CT}$$

* Traces : compositional & based on interactions but they ignore traversed states ...

$CTM \xrightarrow{coin} \bullet \xrightarrow{\overline{coffee}}$ - - - $\overline{tea}$ is possible

$CTM' \xrightarrow{coin} \bullet \xrightarrow{\overline{coffee}}$ - - - $\overline{tea}$ not possible

DEF.

<u>Idea</u>: $P \sim Q$ if

- when $P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} P_2 \cdots \xrightarrow{\alpha_n} P_n$
  then $Q \xrightarrow{\alpha_1} Q_1 \xrightarrow{\alpha_2} Q_2 \cdots \xrightarrow{\alpha_n} Q_n$ and $P_1 \sim Q_1, \dots, P_n \sim Q_n$

- when $Q \xrightarrow{\alpha_1} Q_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} Q_n$
  then $P \xrightarrow{\alpha_1} P_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} P_n$ and $P_1 \sim Q_1 \dots P_n \sim Q_n$

not a definition, but a property of any relation satisfied by many relations between programs...

→ $\emptyset$ empty relation

→ identity $P \sim Q$ iff $P = Q$

* <u>Bisimilarity</u>

<u>Def</u>. A binary relation $R \subseteq Proc \times Proc$ is a <u>bisimulation</u> if for all $P, Q \in Proc$ with $P \, R \, Q$ then

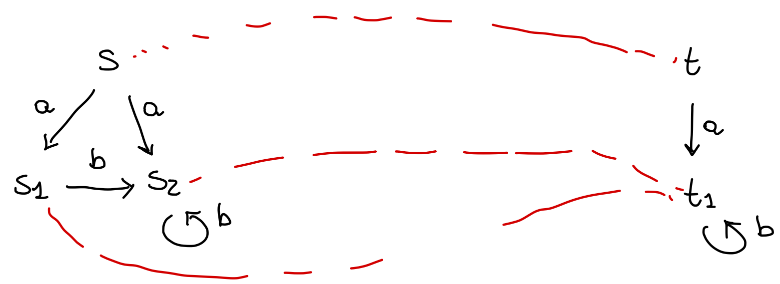
→ for all $P \xrightarrow{\alpha} P'$ there is $Q \xrightarrow{\alpha} Q'$ and $P' \, R \, Q'$

→ for all $Q \xrightarrow{\alpha} Q'$ there is $P \xrightarrow{\alpha} P'$ and $P' \, R \, Q'$

We say P, Q are <u>bisimilar</u> $P \sim Q$ if there is a bisimulation R such that $P \, R \, Q$

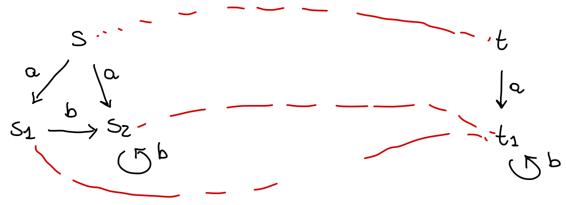
$$\sim \; = \bigcup \{ R \mid R \text{ is a bisimulation} \}$$

<u>Example</u>



$\Uparrow$ $s \sim t$

$R = \{ (s,t), (s_1, t_1), (s_2, t_1) \}$ is a bisimulation



* $s \; R \; t$
  - all transitions of s
    - $s \xrightarrow{a} s_1 \rightsquigarrow t \xrightarrow{a} t_1$ and $s_1 \; R \; t_1$
    - $s \xrightarrow{a} s_2 \rightsquigarrow t \xrightarrow{a} t_1$ and $s_1 \; R \; t_1$
  - all transitions of t
    - $t \xrightarrow{a} t_1 \rightsquigarrow s \xrightarrow{a} s_1$ and $s_1 \; R \; t_1$

* $s_1 \; R \; t_1$
  - all transitions of s_1

    $s_1 \xrightarrow{b} s_2 \rightsquigarrow t_1 \xrightarrow{b} t_1$ and $s_2 \; R \; t_1$
  - all transitions of t_1

    $t_1 \xrightarrow{b} t_1 \rightsquigarrow s_1 \xrightarrow{b} s_2$ and $s_2 \; R \; t_1$

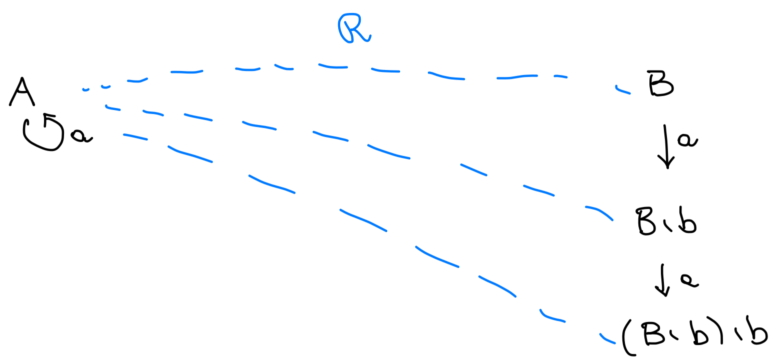
* $s_2 \; R \; t_1$

  same as above

R is bisimulation $\Rightarrow$ $s \sim t$ (but also $s_1 \sim t_1$, $s_2 \sim t_2$)

Example

$A = a.A$ $\qquad\qquad$ $B = (a.B) \setminus b$



infinite states ...

$\vdots$

$R = \{ (A, \underbrace{(((B \setminus b) \setminus b) \dots)}_{K \text{ times}} \setminus b \mid K \in \mathbb{N} \}$

bisimulation

* Non - Bisimilarity ?

$$CTM = coin\ (\overline{coffee}.CTM + \overline{tea}.CTM)$$

$$CTM' = coin\ \overline{coffee}.CTM' + coin.\overline{tea}.CTM'$$

assume $CTM \sim CTM'$

hence there is R bisimulation s.t. $CTM\ R\ CTM'$

note $CTM \xrightarrow{coin} \underbrace{\overline{coffee}.CTM + \overline{tea}.CTM}_{CTM_1}$

then $CTM' \xrightarrow{coin} CTM_1'$ with $CTM_1\ R\ CTM_1'$

2 possibilities

1. $CTM_1' = \overline{coffee}.CTM'$

   since $CTM_1 \xrightarrow{\overline{tea}} CTM$ there should be $CTM_1' \xrightarrow{\overline{tea}}$. No!

2. $CTM_1' = \overline{tea}.CTM_1'$

   since $CTM_1 \xrightarrow{\overline{coffee}} CTM$ there should be $CTM_1' \xrightarrow{\overline{coffee}}$ No!

contradiction

$\Rightarrow$ hence $CTM \not\sim CTM'$