Superconductive Materials

Part 5

BCS Theory

Microscopy understanding of SC, a long story

Failed theories of superconductivity

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Almost half a century passed between the discovery of superconductivity by Kamerlingh Onnes and the theoretical explanation of the phenomenon by Bardeen, Cooper and Schrieffer. During the intervening years the brightest minds in theoretical physics tried and failed to develop a microscopic understanding of the effect. A summary of some of those unsuccessful attempts to understand superconductivity not only demonstrates the extraordinary achievement made by formulating the BCS theory, but also illustrates that mistakes are a natural and healthy part of the scientific discourse, and that inapplicable, even incorrect theories can turn out to be interesting and inspiring.

https://arxiv.org/abs/1008.0447

Microscopy understanding of SC, a long story

Figure 1: Einstein, Bohr, Kronig, Landau, Bloch, and Brillouin.made proposals for microscopic theories of superconductivity prior to the ground breaking experiment by Meissner and Ochsenfeld in 1934.

Figure 2: Between the second world war and the formulation of the BCS theory, unsuccessful attempts to formulate microscopic theories of superconductivity were made by Bardeen, Heisenberg, London, Born, Fröhlich, and Feynman.

https://arxiv.org/abs/1008.0447

BCS Introduction

The phenomenological theories discussed before had the great merit of explaining the several electromagnetic properties of a superconductor in terms of a moderate number of empirical equations combined with the normal metal theory.

Nevertheless no systematic theory of superconductivity explaining the nature of the phenomenon existed up to 1957 when Bardeen, Cooper and Schrieffer (BCS) formulated a microscopical theory based on a "condensation" of couples of electrons into Cooper pairs. Up to this moment the physics of that phenomenon resisted to all efforts of theoretical understanding. A number of talented theoretical physicists attempted to explain by quantum mechanical models the mechanism of no resistance combined with perfect diamagnetism.

The reason was that it was not used the proper tool: the problem was attached in terms of quantum mechanical models of a single-electron motion. But superconductivity is a collective phenomenon: something that results from the cooperation of many atoms together. A single atom of Mercury cannot be superconducting; a cluster of Mercury atoms can.

Roughly speaking superconducting electrons are organized in persistent current circulating on the sample surface just as hypothetical electrons moving in the orbital of a gigantic atom large as the whole sample.

E. Palmieri; The classical superconductivity

Some history about BCS first

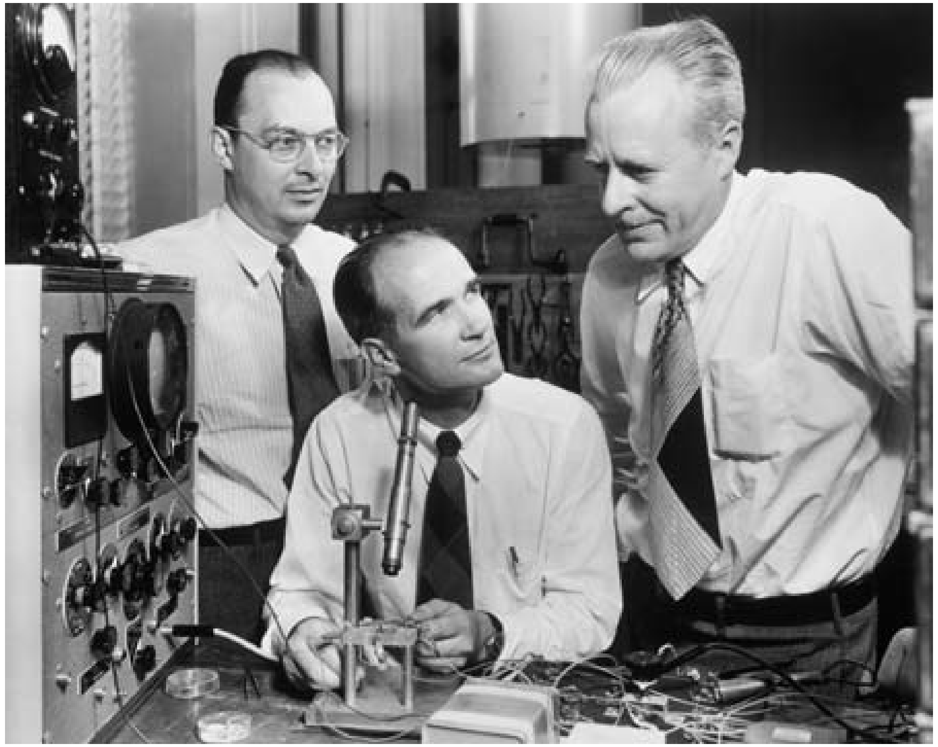
Bardeen during his first academic appointment at the University of Minnesota had an initial crack at finding a theory for SC, work that was published in 1941

Bardeen during the 2nd war years worked at Bell Labs, where **he developed the transistor with Walter Brattain, and William Shockley, winning a Nobel prize in Physics in 1956**

Bardeen left Bell Labs in 1951 and took up a position at the University of Illinois coming back to study SC

After Serin and Rutgers discovered the **isotope effect** built a theory that take in account effect of crystal lattice vibration

Frohlich separately developed a similar theory, but both **did not take in account the Coulomb repulsion**



John Bardeen, Walter Brattain, and William Shockley (seated), in the publicity shot to highlight the invention of the transistor

S. Blundell; Superconductivity, a very short introduction

Some history about BCS first (2)

Bardeen was fascinating by **Pippard's work** that understood that there were **changes** in the **superconductor occurring** in a considerably **thicker layer** beneath the surface, and **not just** within **the thin layer** into which the **magnetic field penetrated**.

By the mid-1950s, Bardeen was beginning to feel that with the right type of theoretical model, it ought to be possible to deduce a superconducting state with a well-defined **coherence length** of the type that Pippard had proposed.

When Bardeen's collaborator **David Pines** left Illinois to work at Princeton, Bardeen hired the post-doc **Leon Cooper** and with a PhD student, **Robert Schrieffer**, began to work intensively on the problem

Some history about BCS first (2)

Cooper breakthrough was the idea to **focus down on just two electrons,** interacting with each other (with all the other electrons ‘frozen’ in place in a so-called ‘Fermi sea’) **instead** of treat the too complicated **full problem of many interacting electron**

Cooper was able to show that an arbitrarily small attraction between electrons can make it cost less energy for the two electrons to pair up together, rather than float as singletons in the Fermi sea

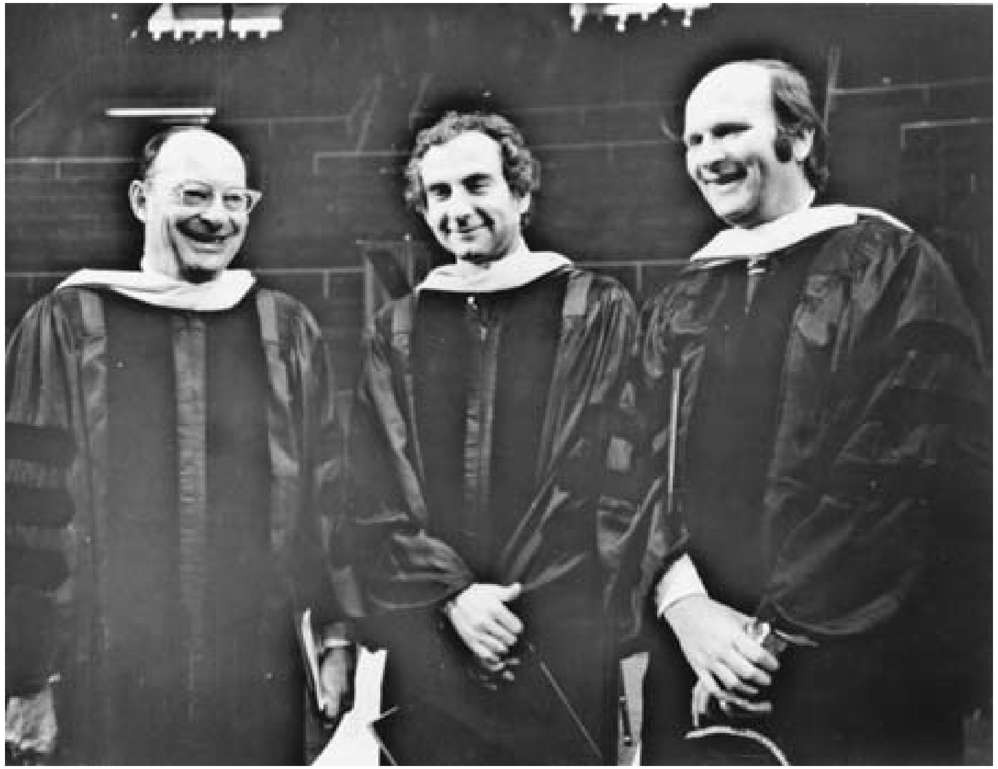
Schrieffer wrote the **BCS wavefunction** (apparently sitting in a New York subway train). In the SC state, **an enormous number of electrons acted in concert**, as if each was part of a larger, inseparable whole. **Contrary to the independent electron model** which was the mainstay of existing solid state physics whereby you could consider electrons one at a time, as if each behaved as an independent entity

Some history about BCS first (3)

Bardeen, Cooper, and Schrieffer found that they were able to systematically **produce predictions of the experimental properties that matched completely with what had been found in real experiments**. The BCS theory worked. Their paper was published in Physical Review in 1957 and was recognized immediately as a masterpiece

The BCS theory explained most of the properties that had been observed in SC up until that time and in 1973 would win its inventors a Nobel Prize.

Bardeen became the first person to win two prizes in the same field



'BCS' (aka John Bardeen, Leon Cooper, and Robert Schrieffer, obligingly standing in their correct order)

S. Blundell
Superconductivity, a very short introduction

What a SC theory must explain in 1957

- Zero Resistance
- Meissner Effect
- Isotope Effect
- Energy Gap
- Specific Heat jump at Tc

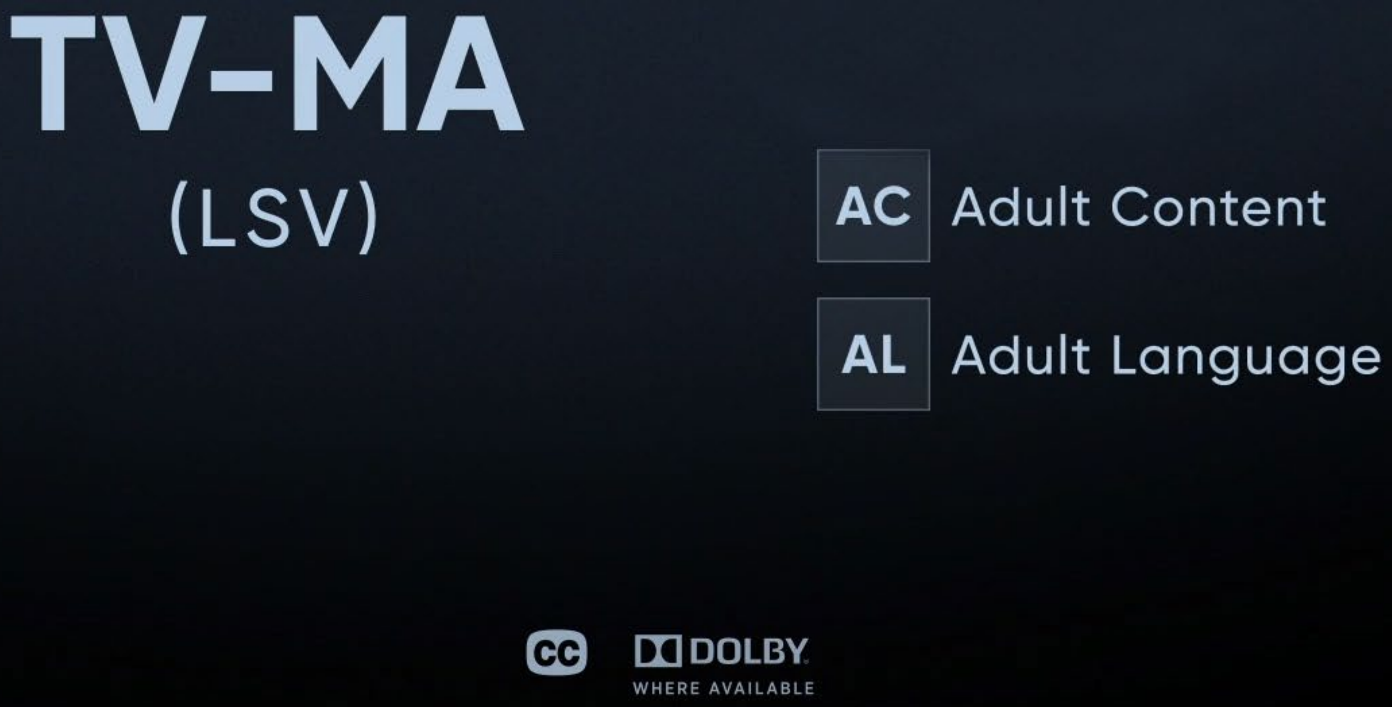
What BCS will stimulate

- Flux Quantization
- Tunnel effect
- Josephson effect

How can we understand an interaction between 2 electrons mediated through a phonon?

Physics have fun to create analogies

A hot analogy



Two particles with a well known repulsion

Woman

Man with headache

Natural repulsion on a bed

Hammock mediated interaction

How we can understand an interaction between 2 electrons mediated through a phonon?

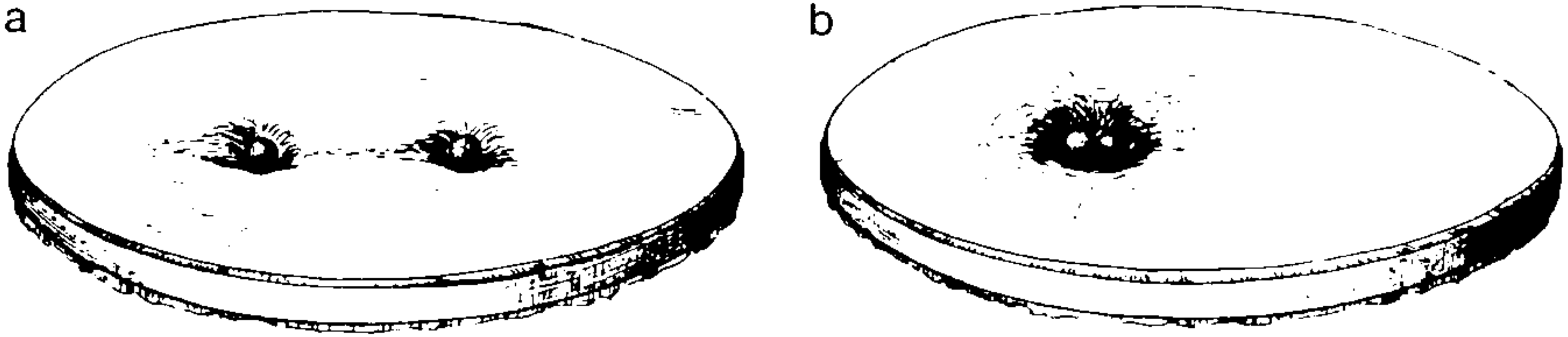
A second analogy

(more common in the books)

Cooper pairing, a mechanical analogy

Two balls in a membrane

We recognize intuitively that the **energy** of the total system **can be lowered** if the **two balls lie in the same hollow**



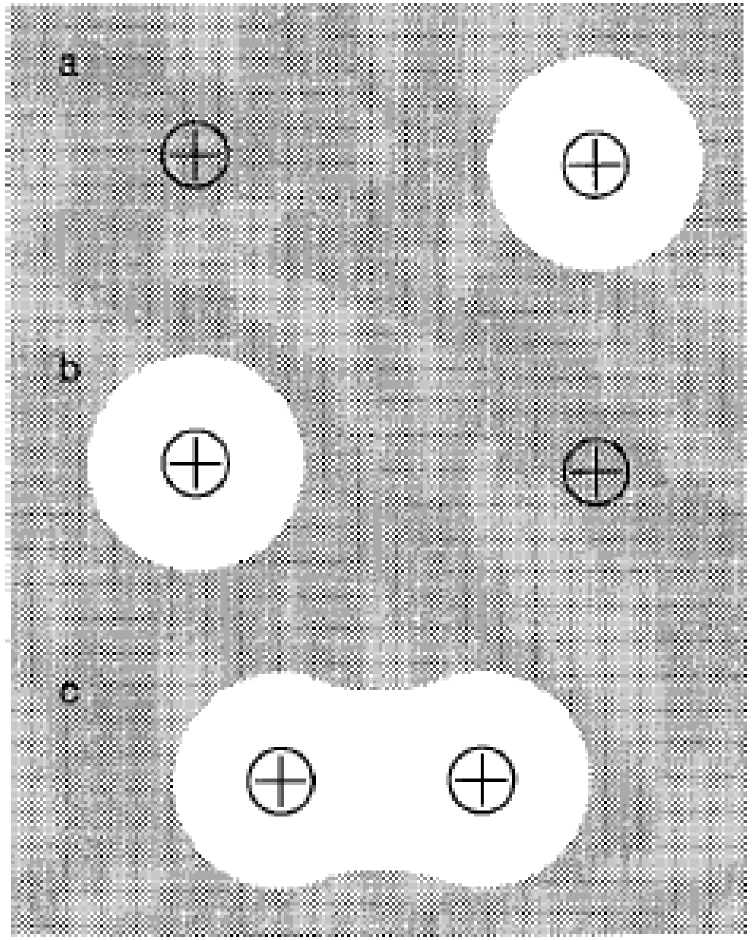
Attraction between two small balls on an elastic membrane.
The configuration (a) is unstable and changes into configuration (b)

Cooper pairing, a quantum mechanic analogy

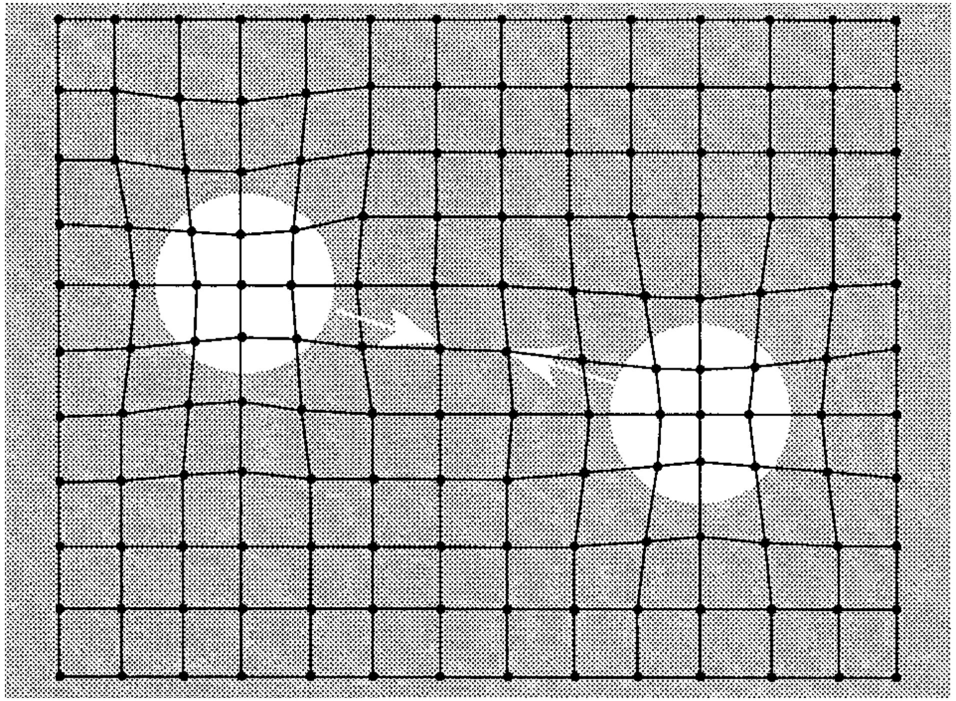
Binding of the H_2 molecule

Energy gain to the exchange of electrons

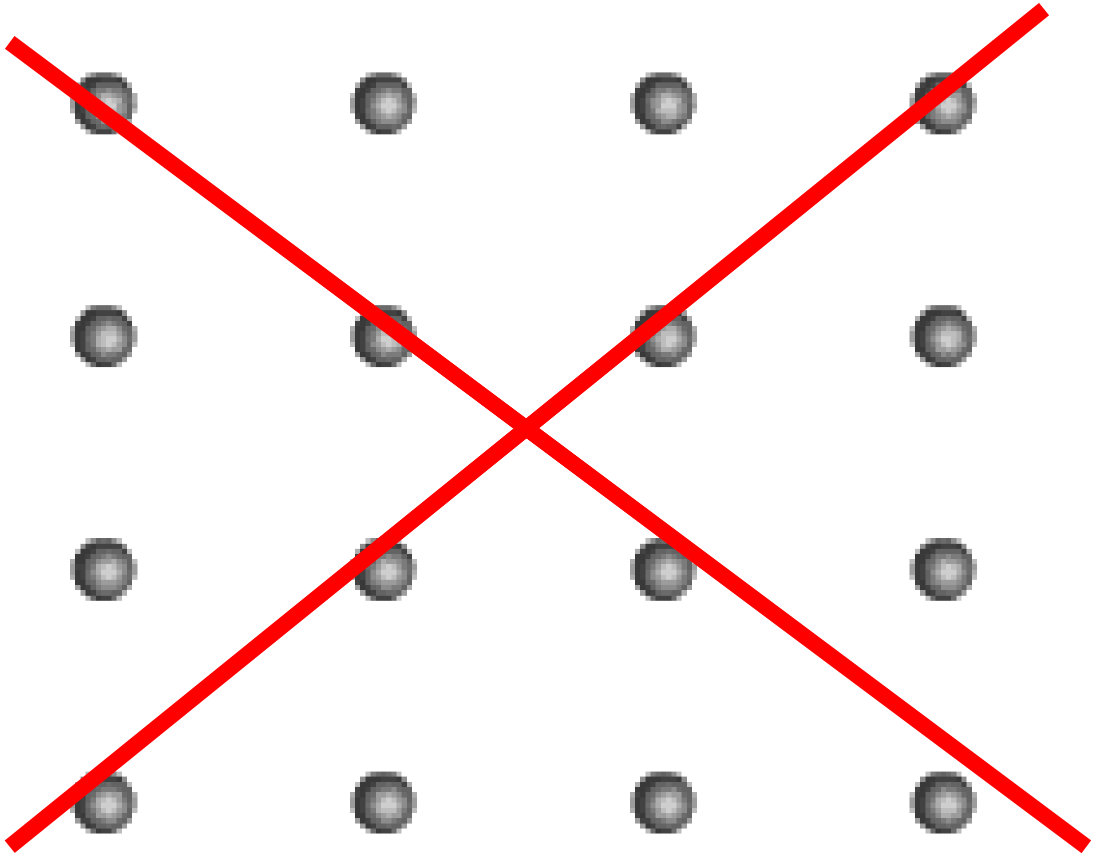
$$\Delta p_x \Delta x = \hbar$$



Cooper pairing, a static model



Polarization of the lattice of atomic ions due to the electrons. Within a static model, this polarization cannot overcompensate the repulsion between the electrons because of their equal charge. It can only strongly reduce the repulsion



COOPER PAIRS have opposite momenta

The dynamic analogy of the skier

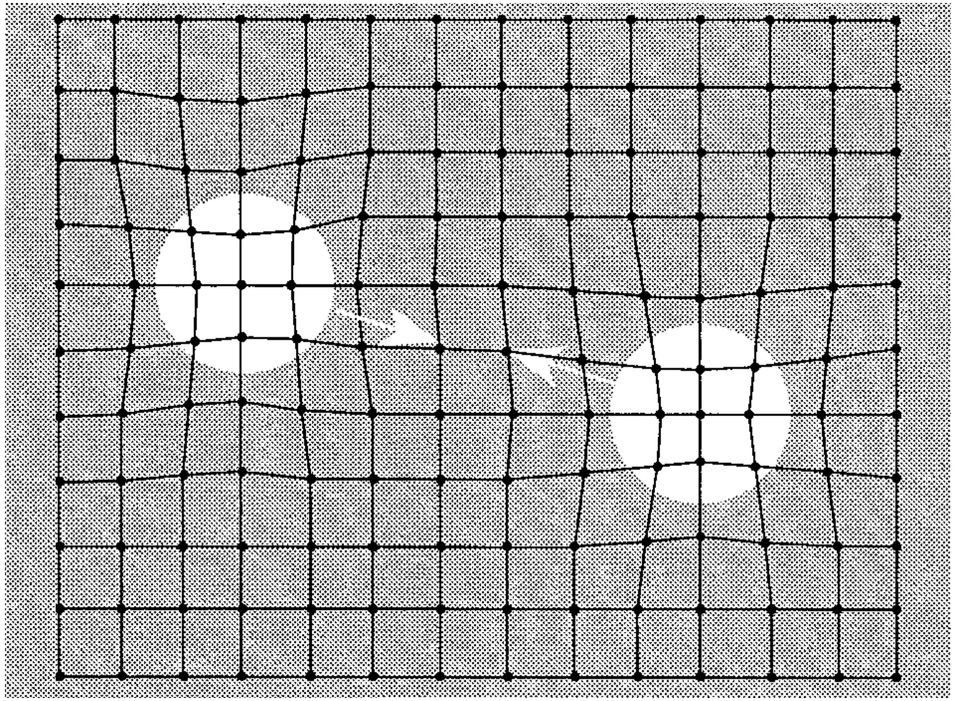
- A cross-country skiing in very deep snow
- Create a sky-track for another skier
- In the analogy the track can be used in both direction

Quantum theory makes a unique choice:

only electrons of opposite momenta form a bound system

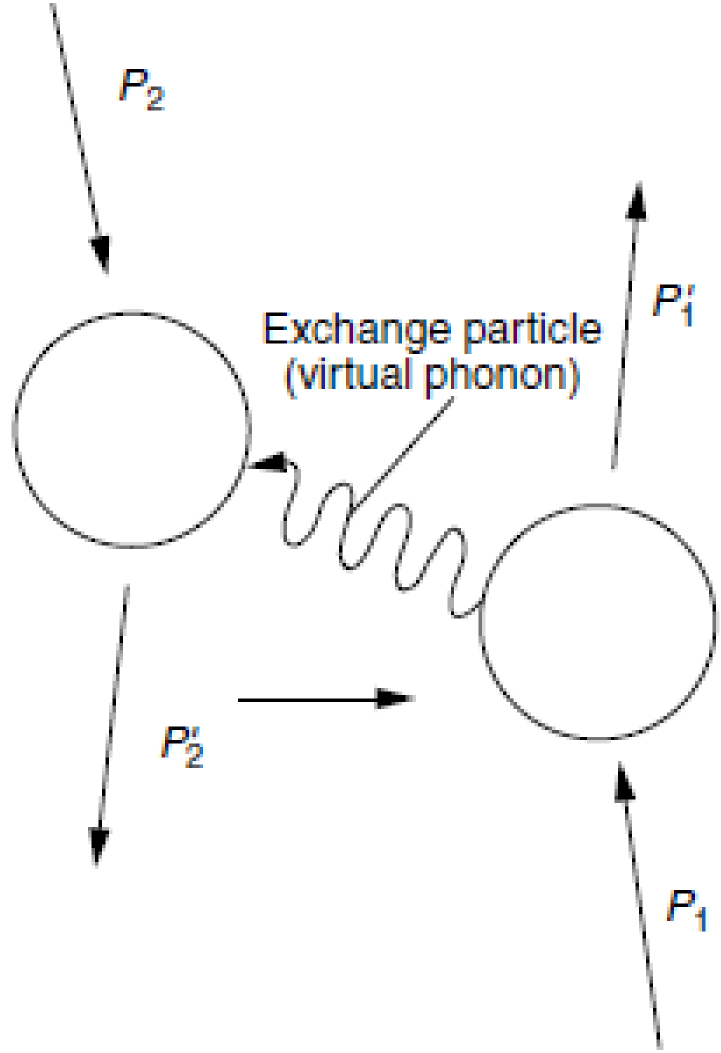
COOPER PAIRS

Cooper pairing, a static model



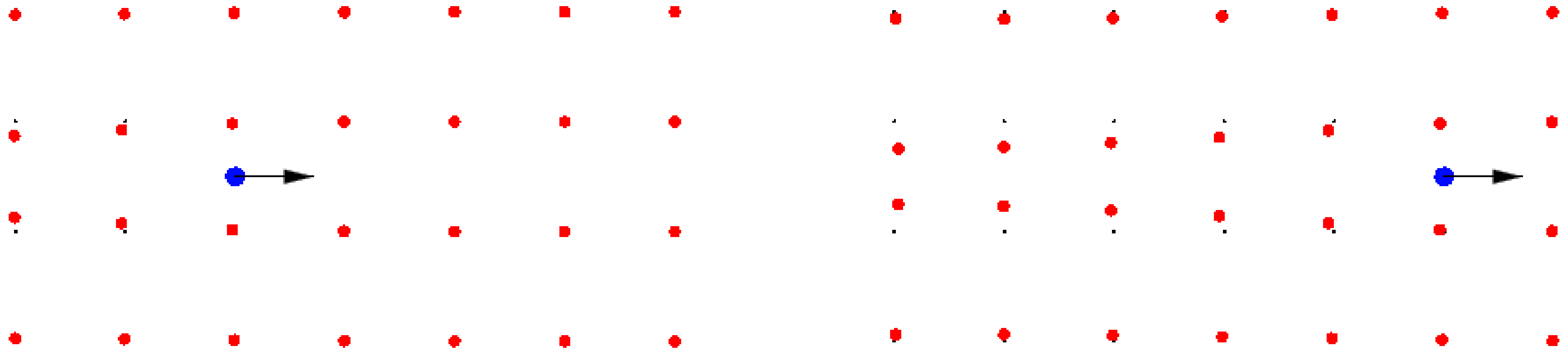
Polarization of the lattice of atomic ions due to the electrons. Within a static model, this polarization cannot overcompensate the repulsion between the electrons because of their equal charge. It can only strongly reduce the repulsion

$$p = \frac{h}{\lambda}$$



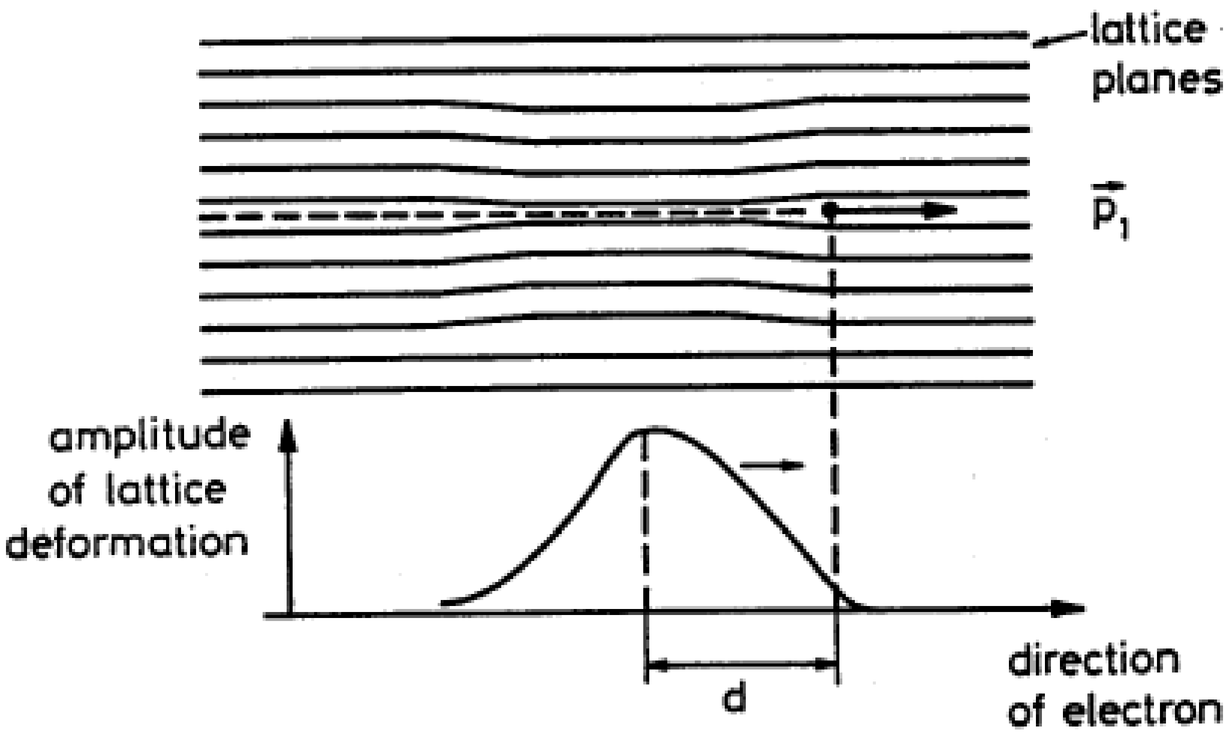
COOPER PAIRS have opposite momenta

Dimension of the lattice deformation



The **shortest response time** corresponds to the highest possible lattice vibration frequency → **Debye frequency ω_D**

Dimension of the lattice deformation



$$d \approx v_F \frac{2\pi}{\omega_D} \approx 100\text{-}1000 \text{ nm}$$

The **shortest response time** corresponds to the highest possible lattice vibration frequency → **Debye frequency $\boldsymbol{\omega_D}$**

First, some basic concepts to remember

In a metal

- **Positively charged ions form a regular crystal lattice**
- **Valence electrons can move through the crystal**
- Coulomb attraction of the positive ions is represented by a potential well with a flat bottom
- The periodic structure is neglected
- **The energy levels comes from Schrodinger equation + Pauli exclusion principle**
- The **electrons** are treated as **independent and non-interacting particles**

In a metal

The **free electron gas** model works very well if we **avoid** the **classical Boltzmann statistics** and we **use the Fermi-Dirac statistics**, based on the Pauli principle

Boltzman $$\frac{1}{2}m_e\boldsymbol{v}^2 = \frac{3}{2}k_B T \quad \Rightarrow \quad E_k \approx 0.025\ eV$$

$$E_k = \frac{3}{5}E_F$$

Fermi-Dirac $$E_F = \frac{\hbar^2}{2m_e}\left(3\boldsymbol{\pi}^2 n\right)^{2/3} \approx 5\ eV \quad \Rightarrow \quad E_k \approx 3\ eV$$

Quantum Mechanics in 1 slide

In a 3D region un the metal, the Schrodinger equation with potential V = 0 and with **periodic boundary conditions** $\psi(x+L, y, z) = \psi(x, y, z)$ etc.

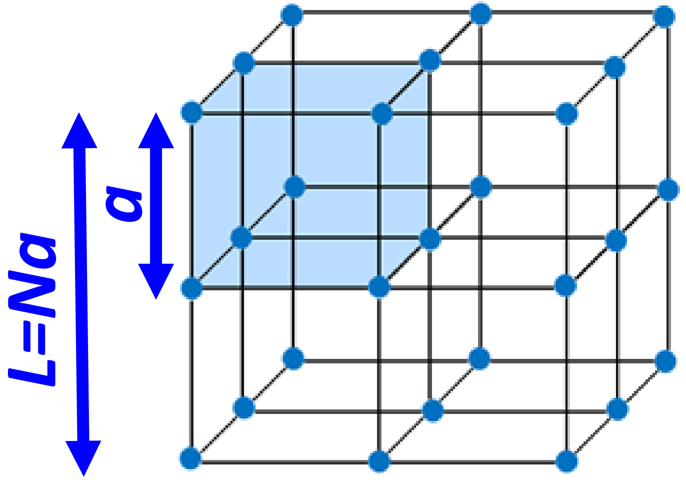
$$\psi(x, y, z) = L^{-3/2} e^{i(k_1 x + k_2 y + k_3 z)}$$

where the **wave vector *k*** $k_j = n_j \frac{2\pi}{L}$ $n_j = 0, \pm 1, \pm 2, \ldots$

Electron momentum $p = \hbar k$

Energy $E = \frac{\hbar^2 k^2}{2m_e}$

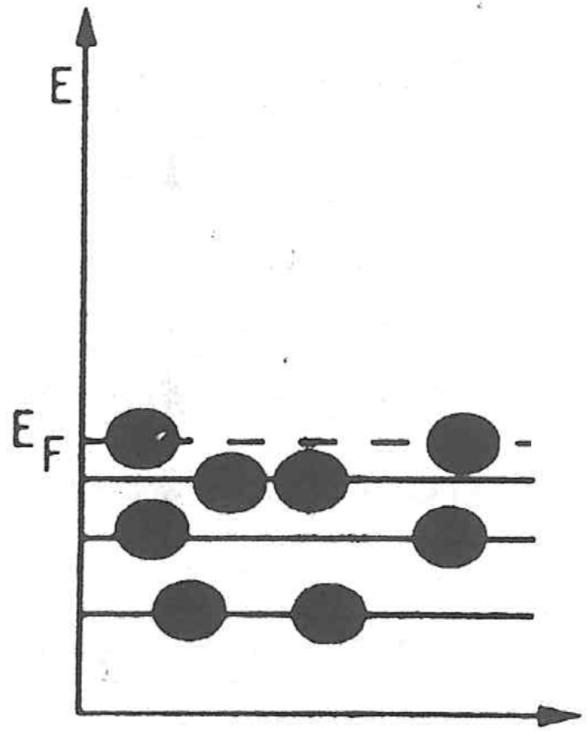
$\hbar = \frac{h}{2\pi} = 1.05 \cdot 10^{-34} Js = 6.58 \cdot 10^{-15} eVs$ *Reduced Plank constant (h-bar)*

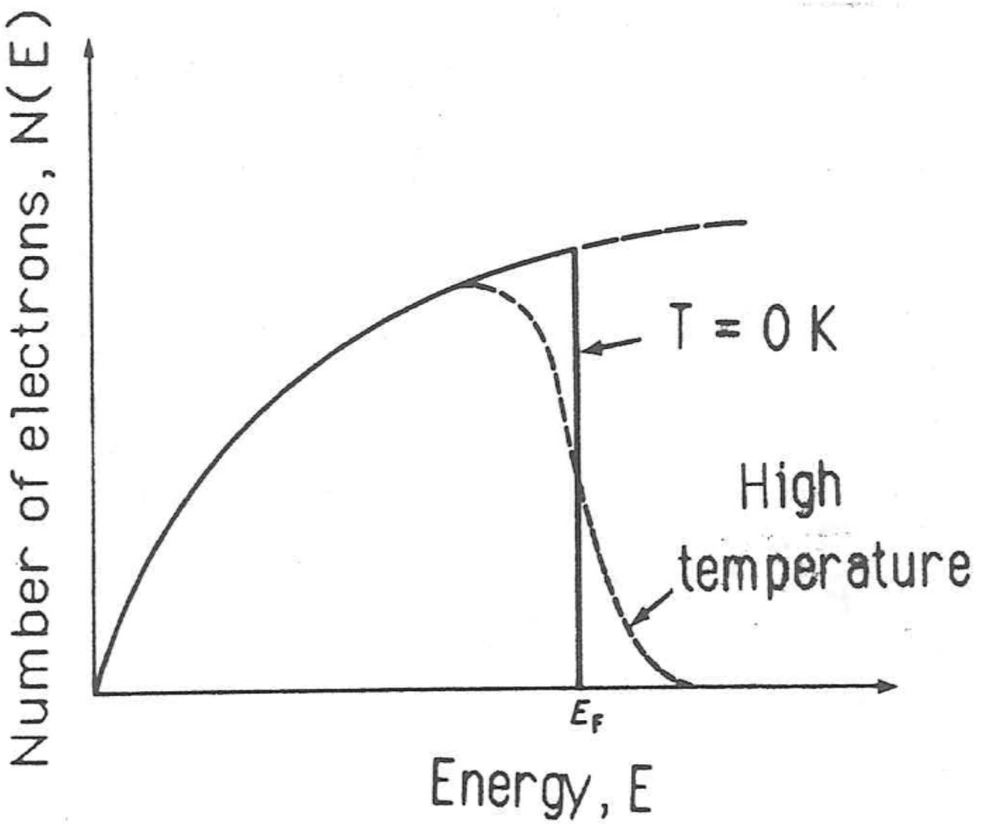


Cristal Lattice

The Fermi Energy

Fermi Energy E_F *is the highest energy level occupied at T → 0 K*

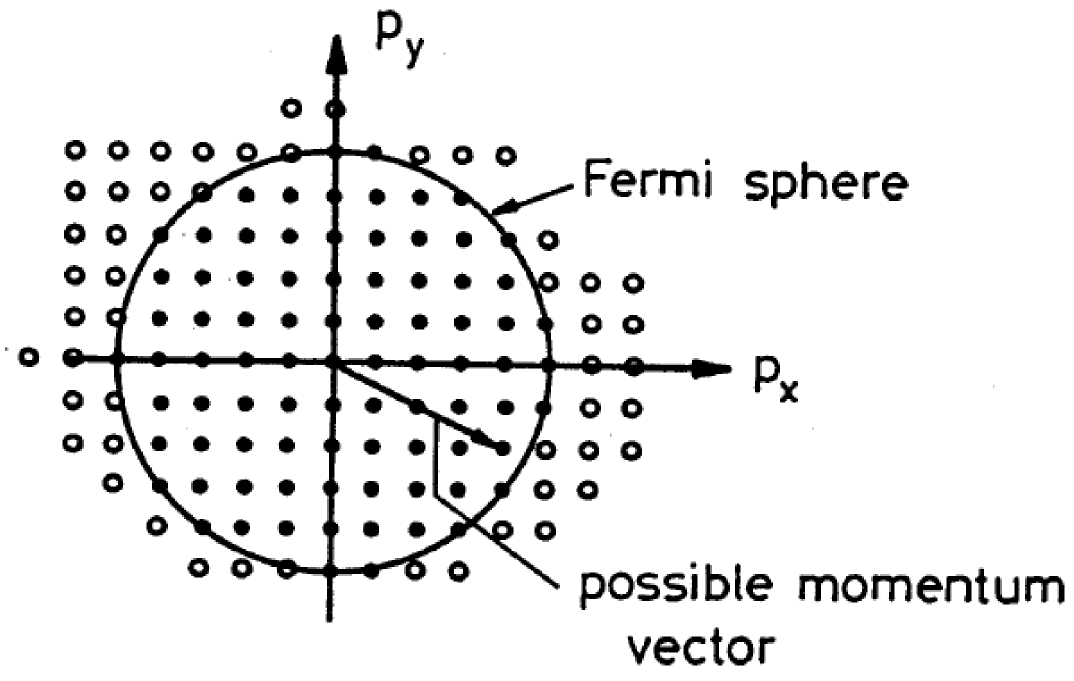




The Fermi Sphere

at T → 0 K ***Fermi momentum p_F*** *is the highest momentum*

$$p_F = \sqrt{2m_e E_F}$$



Fermi velocity v_F

$$v_F = \frac{p_F}{m_e} \approx 10^6 \mathrm{m/s}$$

$$\delta p_x = eE_0 \delta t$$

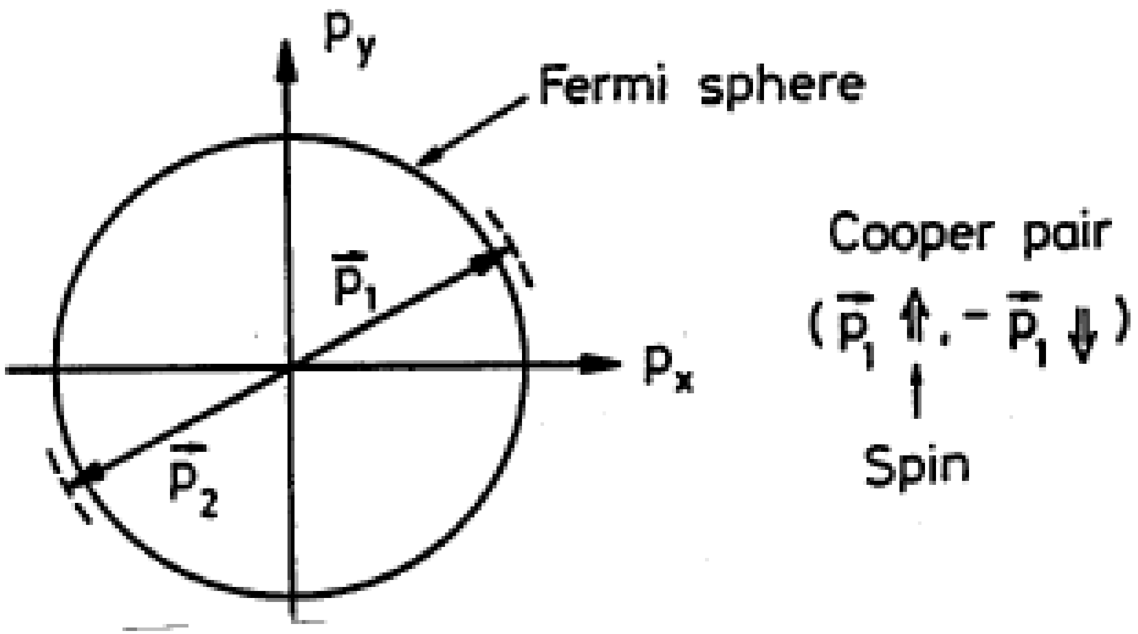
Applied Electric Field

Because of ***Pauli Principle*** *only a fraction of electrons can participate to conduction*

Cooper pairs

Cooper demonstrated that a **couple of 2 electrons with opposite momenta** slightly larger than p_F **shows a weak attractive force**

$$E_{pair} < 2E_f$$



The formation of a Cooper pair

At T=0

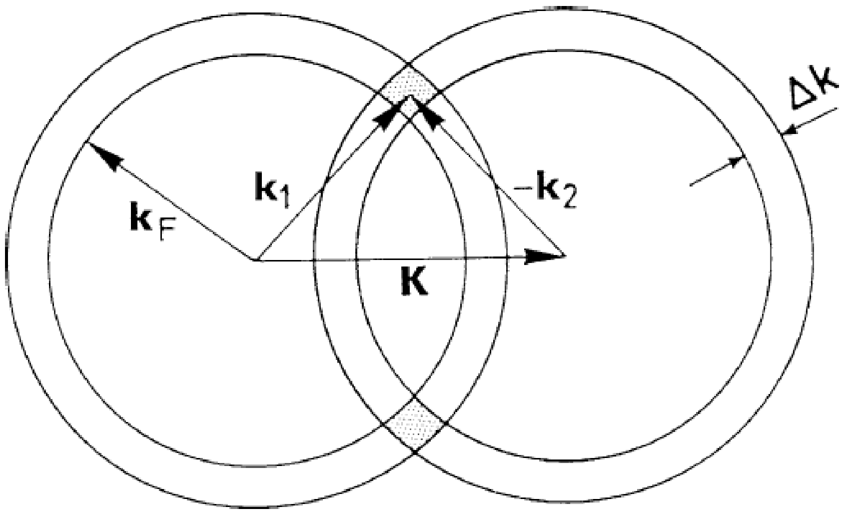
- electrons fill all the energy levels below the Fermi Energy
- all levels above E_F are empty

$$\boldsymbol{p} = \hbar \boldsymbol{k} \qquad \boldsymbol{k_F} = \frac{\sqrt{2m_e E_F}}{\hbar}$$

Fig. 10.8. Representation (in reciprocal space) of electron pair collisions for which $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}'_1 + \boldsymbol{k}'_2 = \boldsymbol{K}$ remains constant. Two spherical shells with Fermi radius k_F and thickness Δk describe the pairs of wave vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. All pairs for which $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{K}$ end in the shaded volume (rotationally symmetric about $\boldsymbol{K}$). The number of pairs $\boldsymbol{k}_1, \boldsymbol{k}_2$ is proportional to this volume in $\boldsymbol{k}$ space and is maximum for $\boldsymbol{K} = 0$

<u>**We add 2 electrons**</u>

The interaction in *k*-space is restricted to a **shell with Energy thickness = $\hbar\boldsymbol{\omega_D}$** → E< E_F forbidden

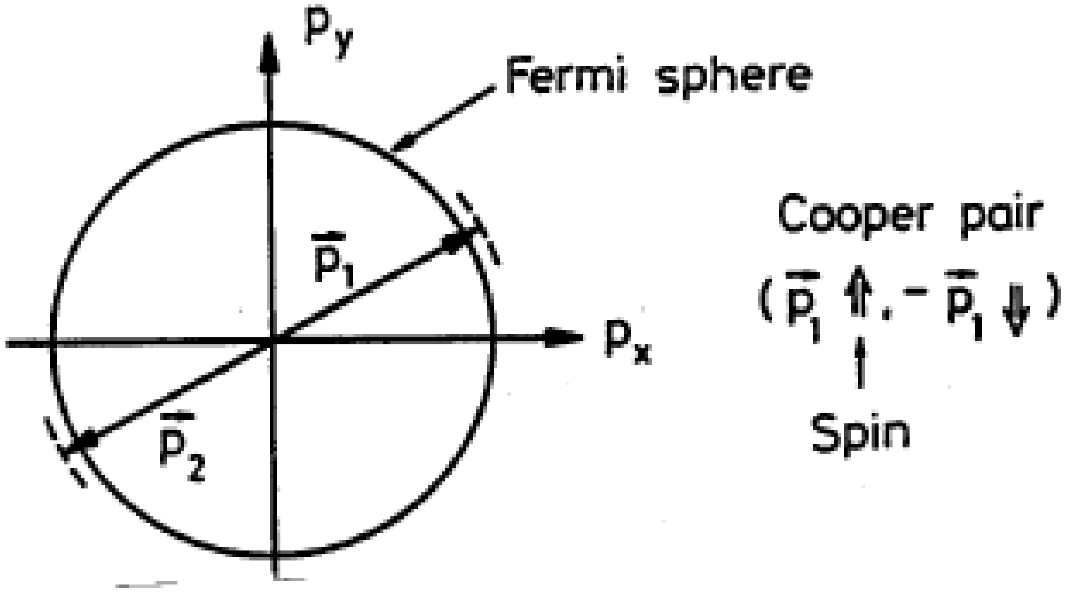


The area (and the number) of energy-reducing phonon exchange processes (and **the strength of the attractive interaction) is maximum for K=0**

The formation of a Cooper pair (2)

So, we can consider the case to add **2 electrons with opposite wavevector**

$$\boldsymbol{k_1} = -\boldsymbol{k_2} \Rightarrow E_1 = E_2 = \frac{\hbar^2 \boldsymbol{k_1^2}}{2m_e}$$



$$\boldsymbol{E_F < E_1 < E_F + \hbar\omega_D}$$

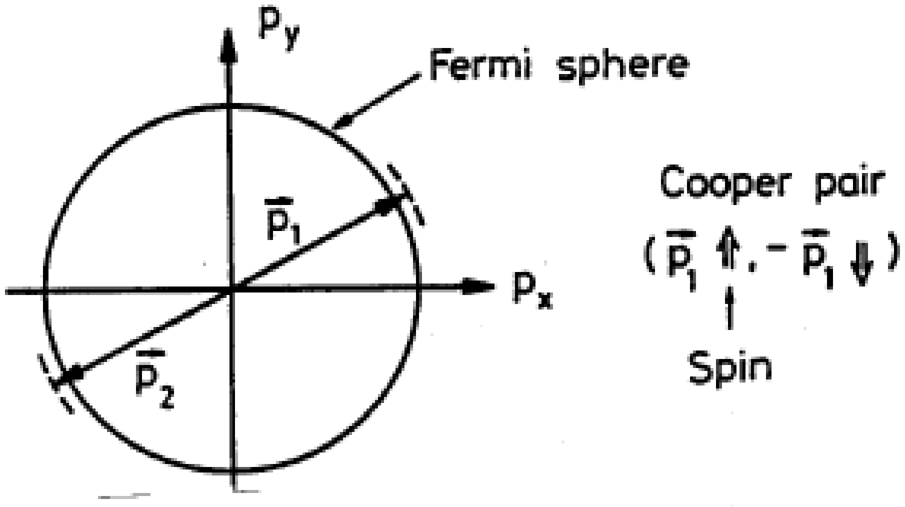
Largest energy quantum of the lattice vibration

The formation of a Cooper pair (3)

$$E_F < E_1 < E_F + \hbar\omega_D$$

It is possible to demonstrate that:

$$E_{pair} = 2E_F - \delta E < 2E_F$$



The problem is reduced to the solution of the Schrödinger equation:

$$-\frac{\hbar^2}{2m_e}(\nabla_1^2 + \nabla_2^2)\psi(\boldsymbol{r_1}, \boldsymbol{r_2}) + V(\boldsymbol{r_1}, \boldsymbol{r_2})\psi(\boldsymbol{r_1}, \boldsymbol{r_2}) = E_{pair}\psi(\boldsymbol{r_1}, \boldsymbol{r_2})$$

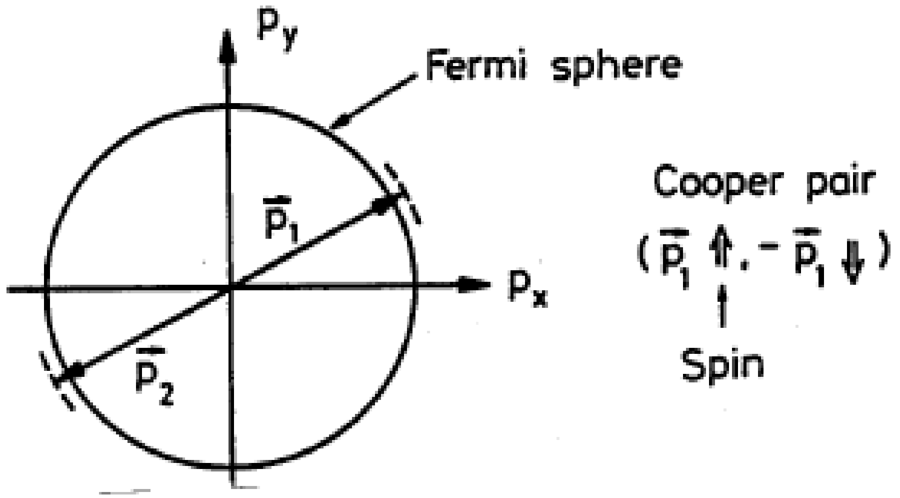
The formation of a Cooper pair (4)

$$-\frac{\hbar^2}{2m_e}(\nabla_1^2 + \nabla_2^2)\psi(\boldsymbol{r_1},\boldsymbol{r_2}) + V(\boldsymbol{r_1},\boldsymbol{r_2})\psi(\boldsymbol{r_1},\boldsymbol{r_2}) = E_{pair}\psi(\boldsymbol{r_1},\boldsymbol{r_2})$$

Solving the equation you can find:

$E_{pair} = 2E_F - \delta E < 2E_F$

$$\boldsymbol{\delta E = 2\hbar\omega_D e^{\left(-\frac{2}{V_0\mathcal{N}(E_F)}\right)}}$$



$\boldsymbol{\omega_D}$ Debay Frequency

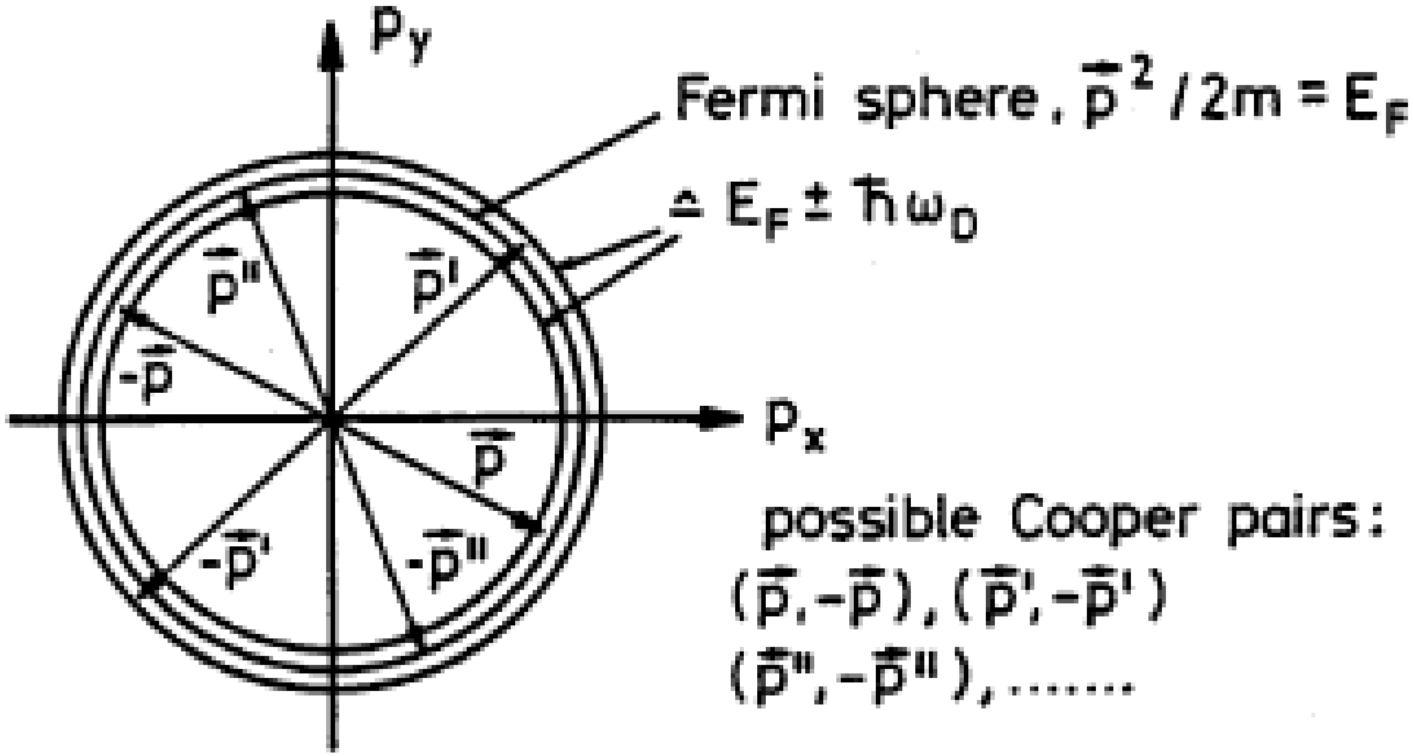
$\boldsymbol{\mathcal{N}(E_F)}$ Density of density of single-electron states of a given spin orientation at E = E_F

$\boldsymbol{V_0}$ electron-lattice interaction potential

Peter Schmuser, Superconductivity
CERN Accelerator School (2004)
https://cds.cern.ch/record/503603

The formation of a Cooper pair (5)

Only a **small fraction of the electrons** can be **paired via phonon exchange**, namely those in a shell of thickness $\mathbf{E_F \pm \hbar\omega_D}$



Energy of the condensed state

The **energy** that **phonons can exchange with electrons** is in the order of:

$$\hbar\omega \leq \hbar\omega_D \ll E_F$$

$$\delta E = 2\Delta = 2\hbar\omega_D e^{\left(-\frac{2}{V_0 \mathcal{N}(E_F)}\right)}$$

$2\hbar\omega_D$: 10^{-2} eV

$V_0\mathcal{N}(E_F)$: 10^{-2} eV

COOPER PAIR DIMENSIONS

Energy

$2\Delta \sim 10^{-3} - 10^{-4}$ eV

Life time

$\tau \sim \hbar/\Delta \sim 10^{-11} - 10^{-12}$ s

Speed

$v_F = \hbar k_F/m \sim 10^{-5}$m/s

Distance travelled in the opposite direction?

$\xi \sim v_F \tau \sim 10^{-6} - 10^{-7}$m

Coherence lenght

Energy scale

$\delta E = 2\Delta$	$\hbar\omega\_D$	E_F
$10^{-4} - 10^{-3}$ eV	$0.01 - 0.02$ eV	10 eV
$T_c \sim$ 1-10 K	$k_B\Theta_D \sim$ 100-400 K	$T_F \sim 10^4 - 10^5$ K

Coulomb repulsion

$$U_{e\text{-}e} = \frac{e^2}{4\pi\varepsilon_0 r} \sim 1\text{–}10 \text{ eV}$$

How can works Cooper condensation?

Energy scale

$\delta E = 2\Delta$	$\hbar\omega\_D$	E_F
$10^{-4} - 10^{-3}$ eV	$0.01 - 0.02$ eV	10 eV
$T_c \sim$ 1-10 K	$k_B \Theta_D \sim$ 100-400 K	$T_F \sim 10^4 - 10^5$ K

Coulomb repulsion

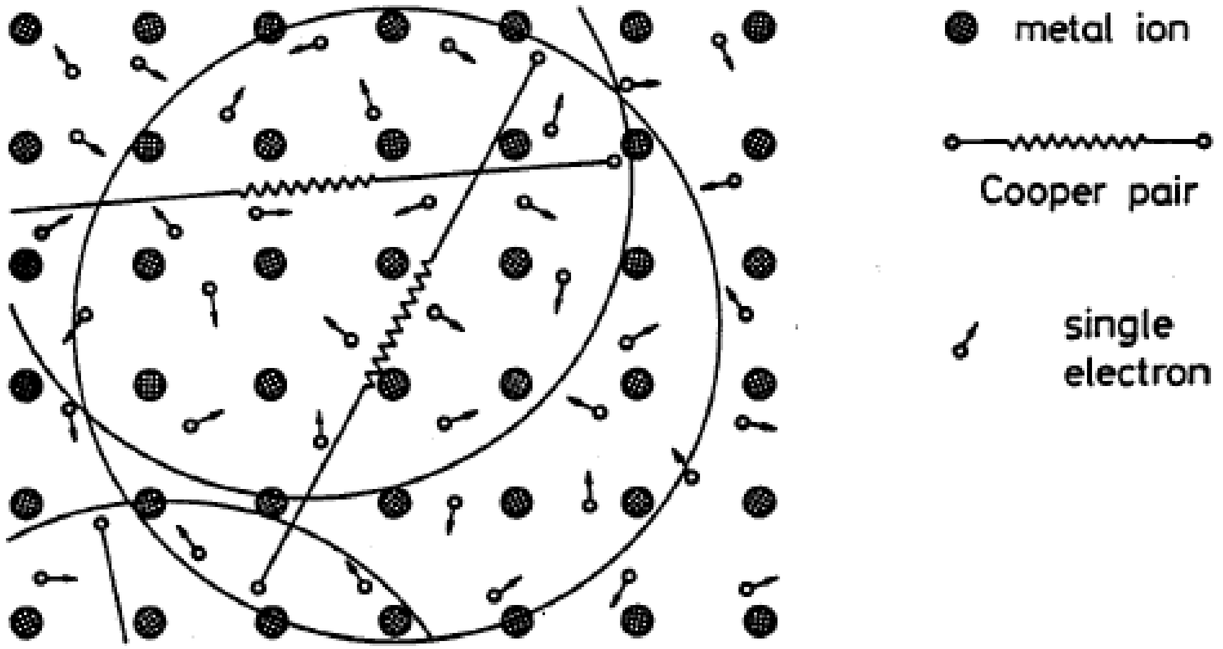
$$U_{e\text{-}e} = \frac{e^2}{4\pi\varepsilon_0 r} \sim 1\text{–}10 \text{ eV}$$

Average distance conduction electrons

$$r \sim 2\text{–}5 \text{ Å}$$

Over-screening

Since the **exchange interaction** occurs when the electrons are at a **great distance** (of the order of the coherence length $\xi \sim 10^{-6}$ m) the **coulomb repulsive interaction is strongly suppressed** ($U_{e\text{-}e} \sim 10^{-4}$-$10^{-3}$ eV $\sim \Delta$)

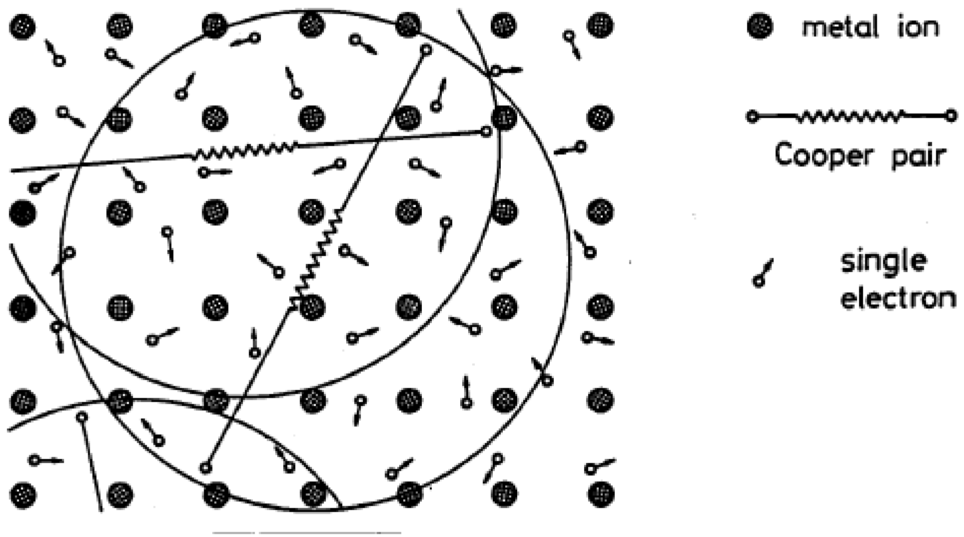


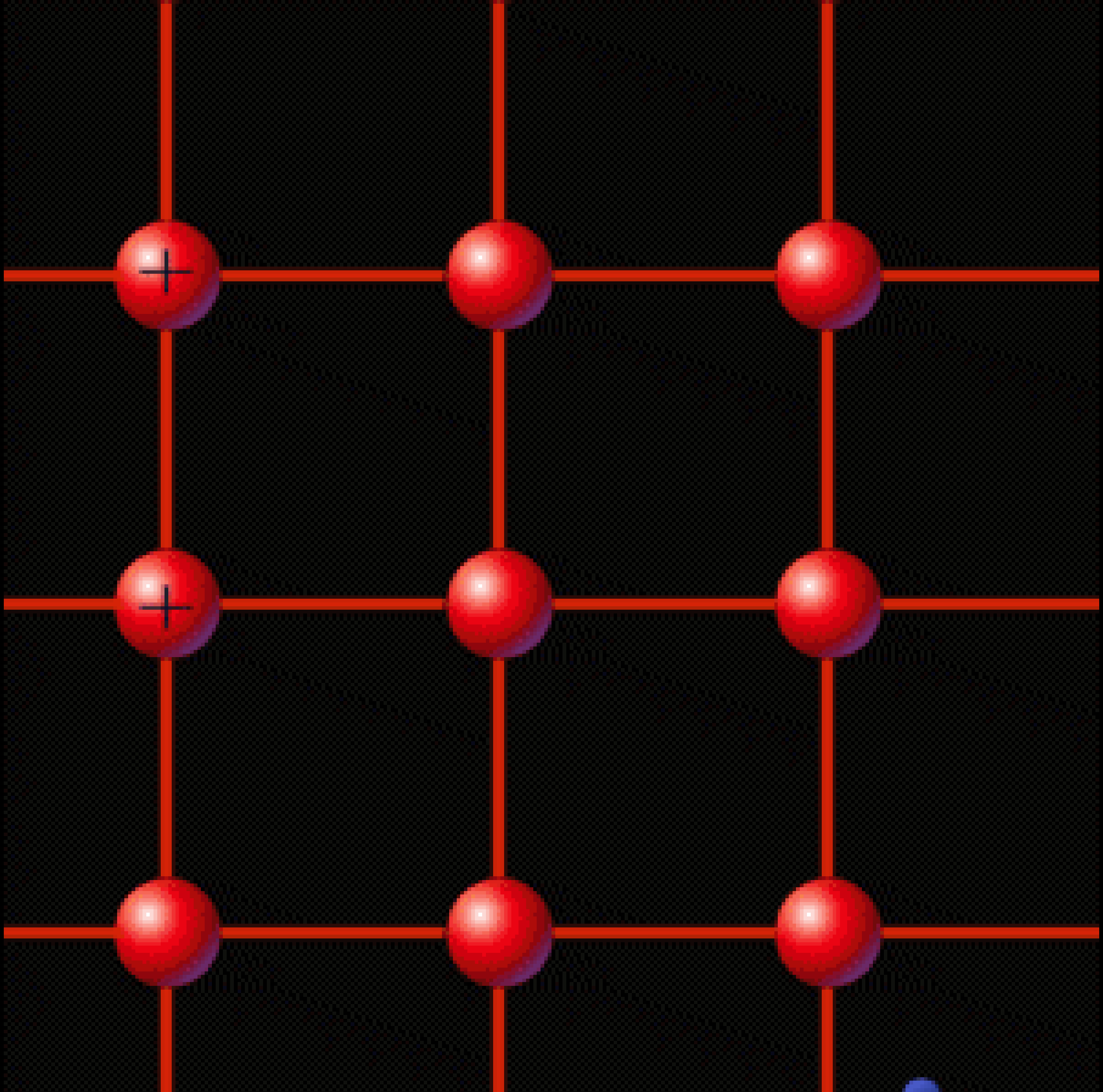
Cooper pairs overlap

Cooper pairs in a SC overlap each other

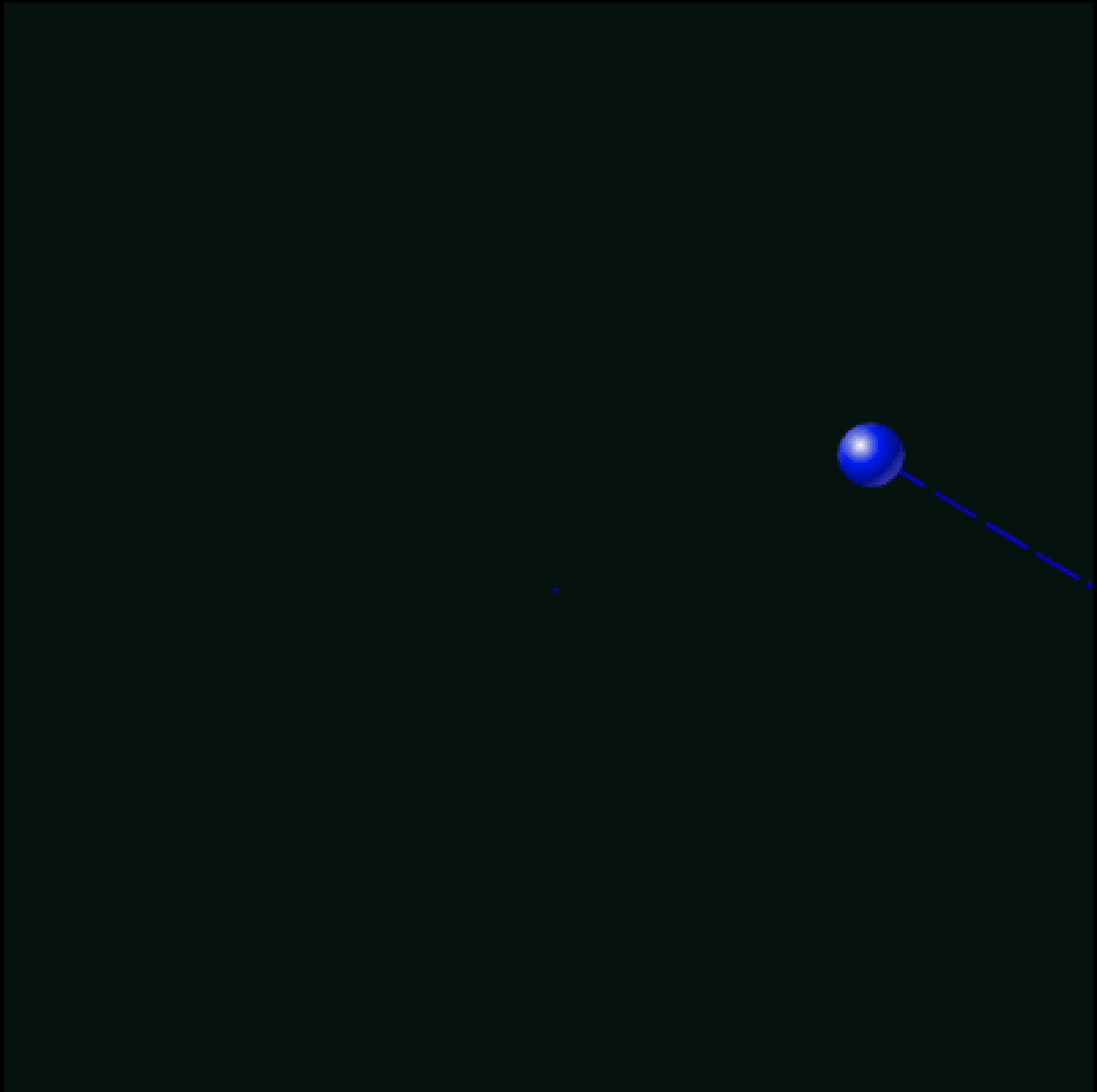
Within the space of a **single pair** there exist **10^6 to 10^7 other electrons**

The **strong overlap** is an important prerequisite of the BCS theory because the **Cooper pairs must change their partners frequently** in order to provide a continuous binding





Marina Putti,
University of Genoa



Marina Putti,
University of Genoa

BCS Theory

Many-body theory:

the wave function is representative of an N-electron system

The spatial part of the wave function is built as a **product of wave functions of electron pairs with zero pulse**

$$\psi_N(\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_N) = \psi(\boldsymbol{r}_1 - \boldsymbol{r}_2)\psi(\boldsymbol{r}_3 - \boldsymbol{r}_4)\dots\psi(\boldsymbol{r}_{N-1} - \boldsymbol{r}_N)$$

BCS Theory (2)

Around the surface of the Fermi sphere, the **Cooper pairs** constitute a highly peculiar state, in which, within a certain interval around the Fermi energy E_F, the pair states $|\boldsymbol{k}\uparrow, -\boldsymbol{k}\downarrow\rangle$ are **simultaneously unoccupied with probability $|u_k|^2$ and occupied with probability $|v_k|^2$**

$$|\boldsymbol{v_k}|^2 = \frac{1}{2}\left[1 - \frac{\boldsymbol{\varepsilon_k - E_F}}{\sqrt{\boldsymbol{\Delta_0^2 + (\varepsilon_k - E_F)^2}}}\right]$$

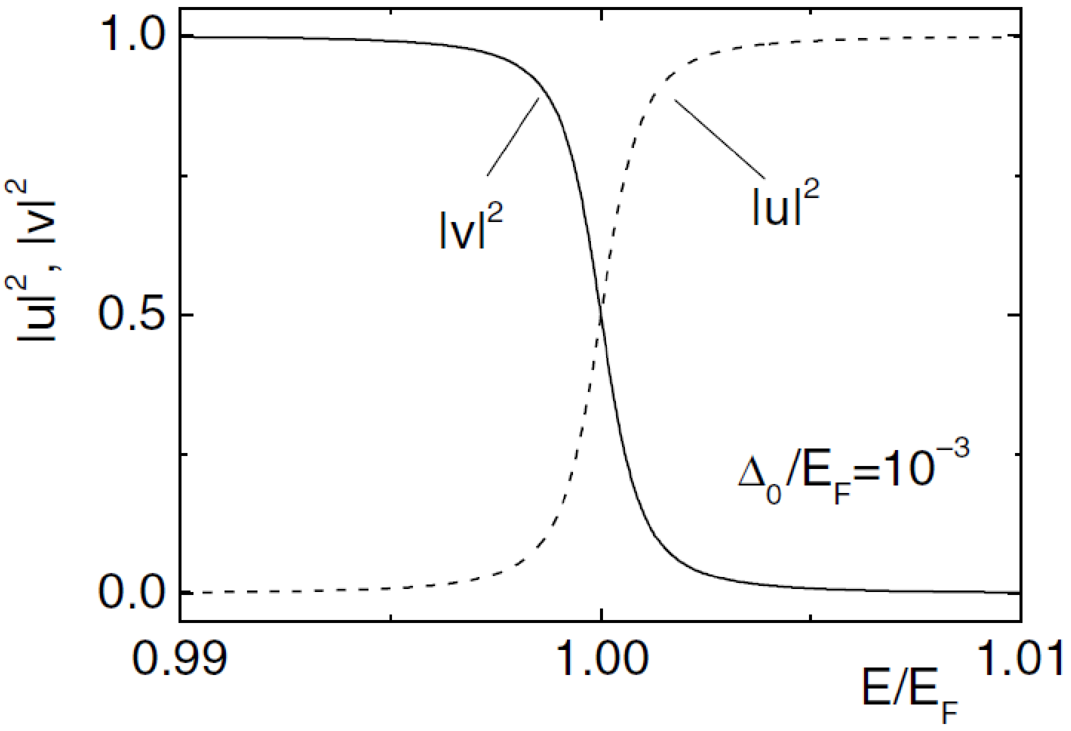
$$|\boldsymbol{u_k}|^2 = \frac{1}{2}\left[1 + \frac{\boldsymbol{\varepsilon_k - E_F}}{\sqrt{\boldsymbol{\Delta_0^2 + (\varepsilon_k - E_F)^2}}}\right]$$

$\boldsymbol{\varepsilon_k}$ energy of the individual electrons in the absence of the interaction V

$$\boldsymbol{\Delta = \sum_k u_k v_k} \Rightarrow \boldsymbol{\Delta \neq 0} \text{ only for } \boldsymbol{\varepsilon_k \pm \hbar\omega_D}$$

$$\boldsymbol{\Delta = \Delta_0\, e^{i\varphi}} \Rightarrow \boldsymbol{|\Delta|^2 = \Delta_0^2}$$

BCS Theory (3)



$\boldsymbol{\Delta}_0$ = 1 meV

E_F = 1 eV

The quantity **Δ depends on all states *k*** because of the product $u_k v_k$

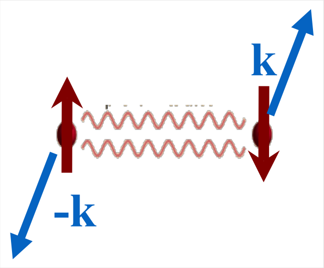
This shows that **all Cooper pairs are collectively connected** with each other

The ratio $\boldsymbol{\Delta}_0/E_F$ is equal to 10^{-3}

Only about **0.1 percent of all electrons participate in the SC**

Fundamental state in BCS theory

Overlap of states $|\boldsymbol{k}\uparrow, -\boldsymbol{k}\downarrow\rangle$ ➡ ***Cooper pairs***



Energy of the single e⁻ in a Cooper pair

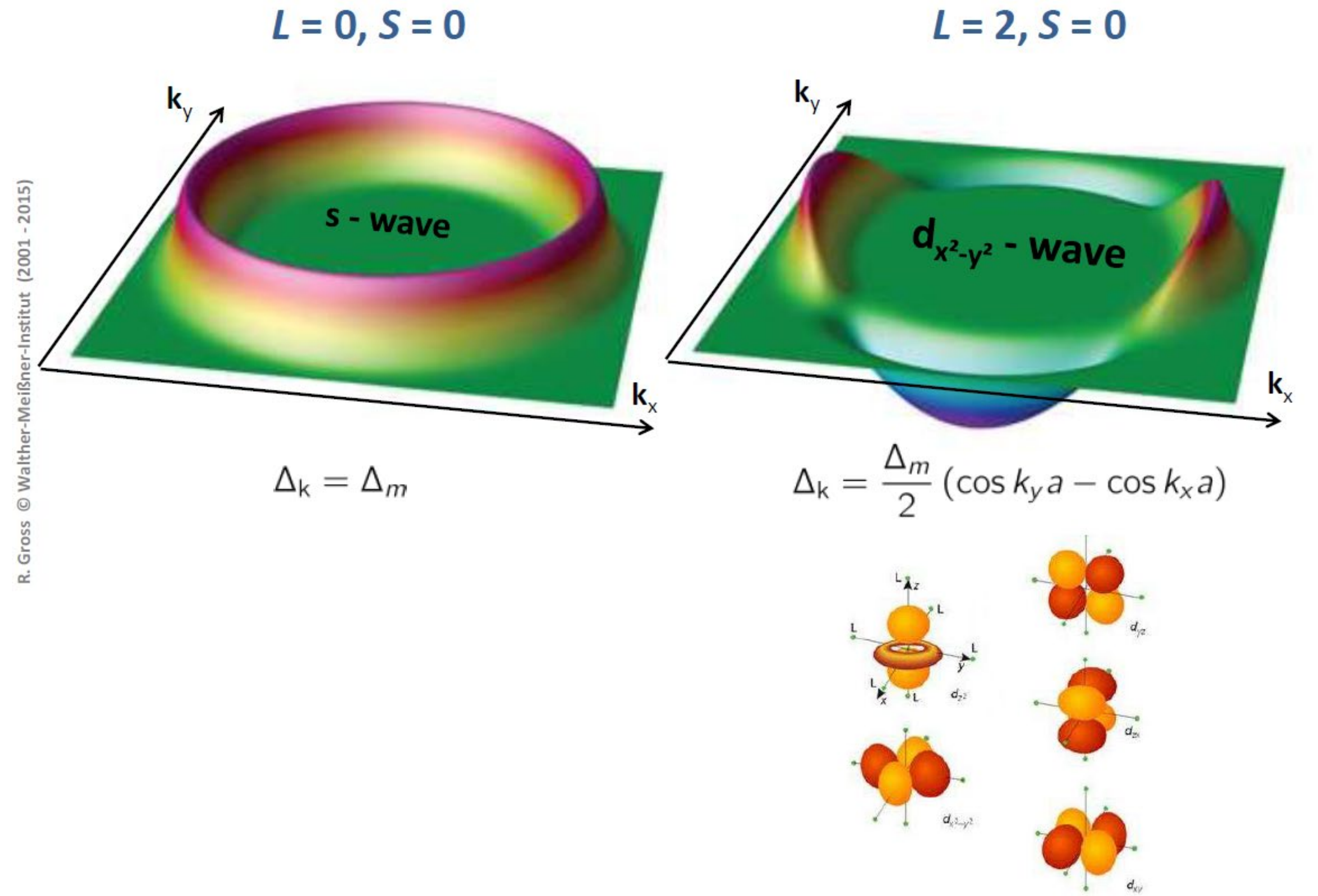
$$\Delta(0) = 2\hbar\omega_D e^{\left(-\frac{1}{VN_n(E_F)}\right)}$$

ω_D Debay Frequency

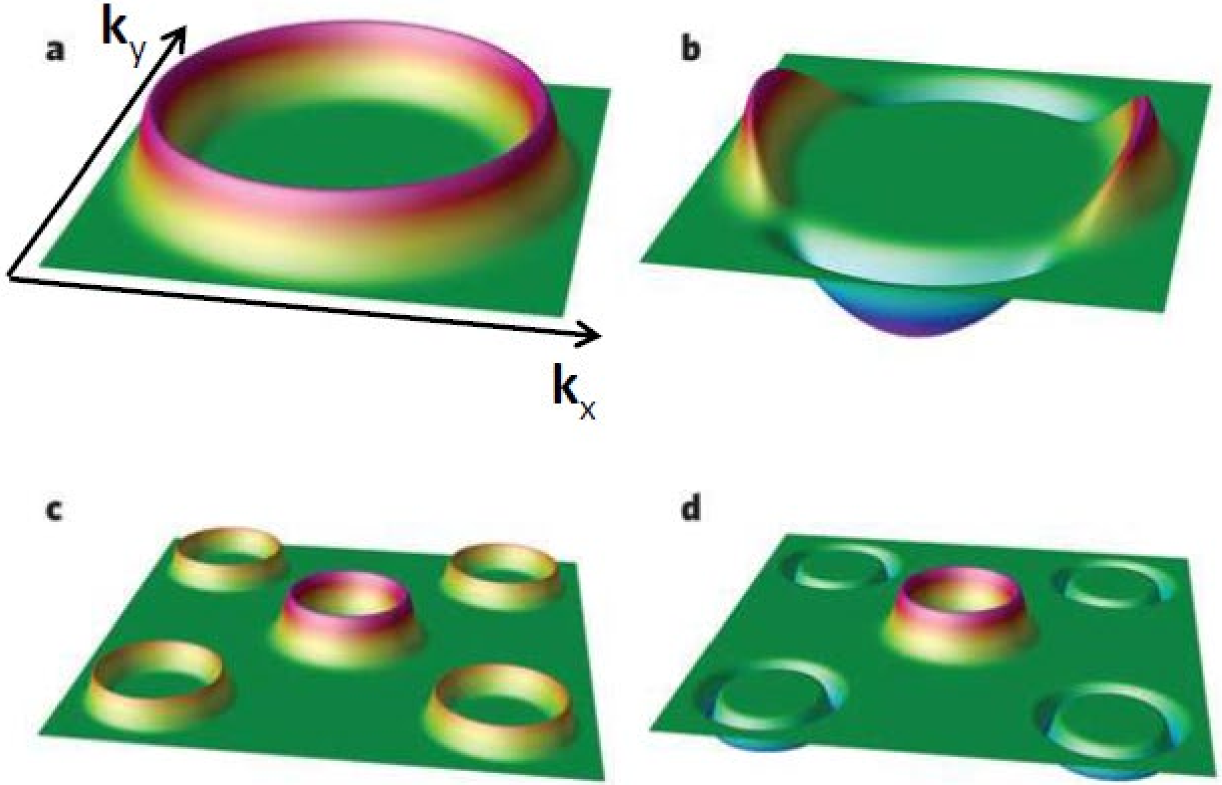
$\mathcal{N}(E_F)$ Density of single-electron states of a given spin orientation at $E = E_F$

V_0 electron-lattice interaction potential

Symmetry of Pair Wavefunction



Symmetry of Pair Wavefunction



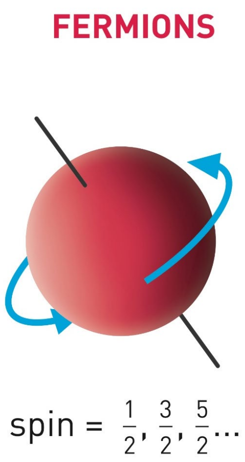
(**a**) *s*-wave, e.g. in aluminium

(**b**) *d*-wave, e.g. in copper oxides

(**c**) two-band *s*-wave with the same sign, e.g. in MgB_2

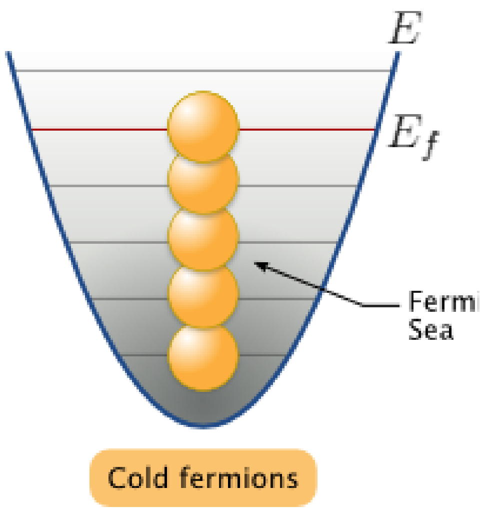
(**d**) an $s_{\pm}$-wave, e.g. in iron-based SC

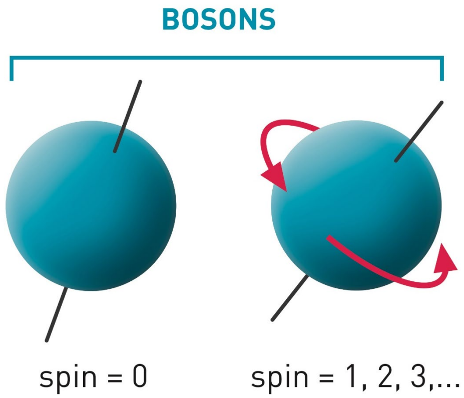
Fermions and Bosons

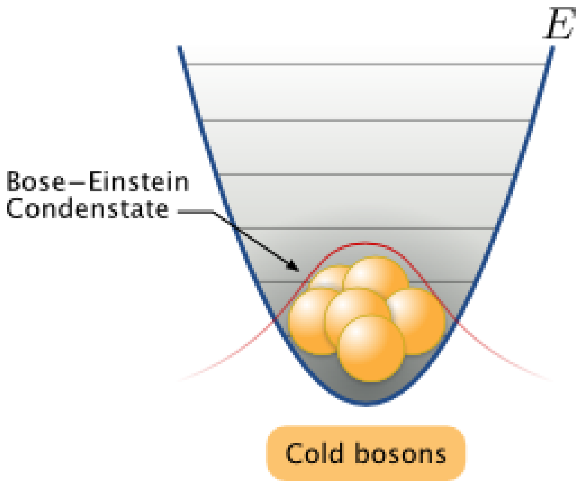


Elementary particles: electrons, muons, neutrinos, quarks

Composite fermions: protons, neutrons, baryon (triplets of quarks), nucleus







Elementary particles: photons, gluons, higgs boson

Composite bosons: mesons (quark-antiquarks pairs)

Nuclei of even mass number: deuterium, He-4, Pb-208)

Quasiparticles: phonons, plasmons

Cooper pairs

Cooper pairs as bosons

All Cooper pairs have therefore the **same momentum P = 0**

All Cooper pairs **occupy exactly the same quantum state**

Cooper pairs **can be described by a macroscopic wave function**

In analogy with the light wave in a laser in which the photons **are all in phase and have the same wavelength, direction and polarization**

The reason why Cooper pairs are allowed and even prefer to enter the same quantum state is that they behave as **Bose particles with spin 0**

Difference from other bosons

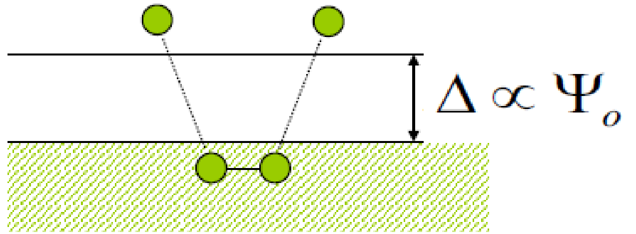
Very extended objects

Cooper pairs **exist only in the BCS ground state** and there is no excited state

⬇

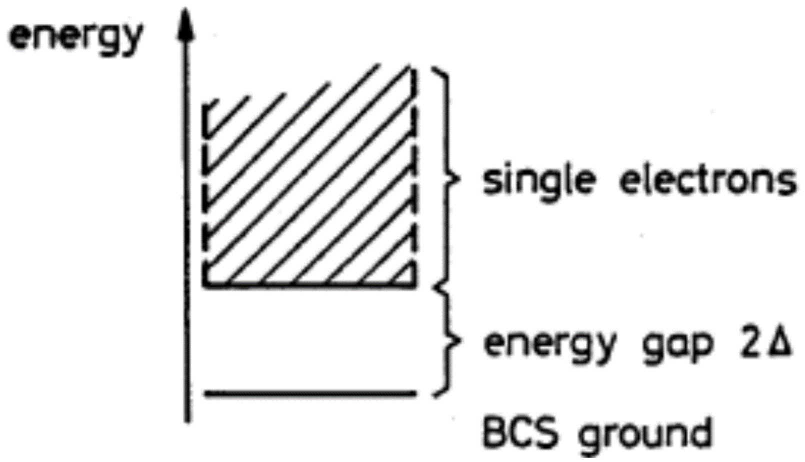
An **excitation is equivalent to breaking them up** into single electrons

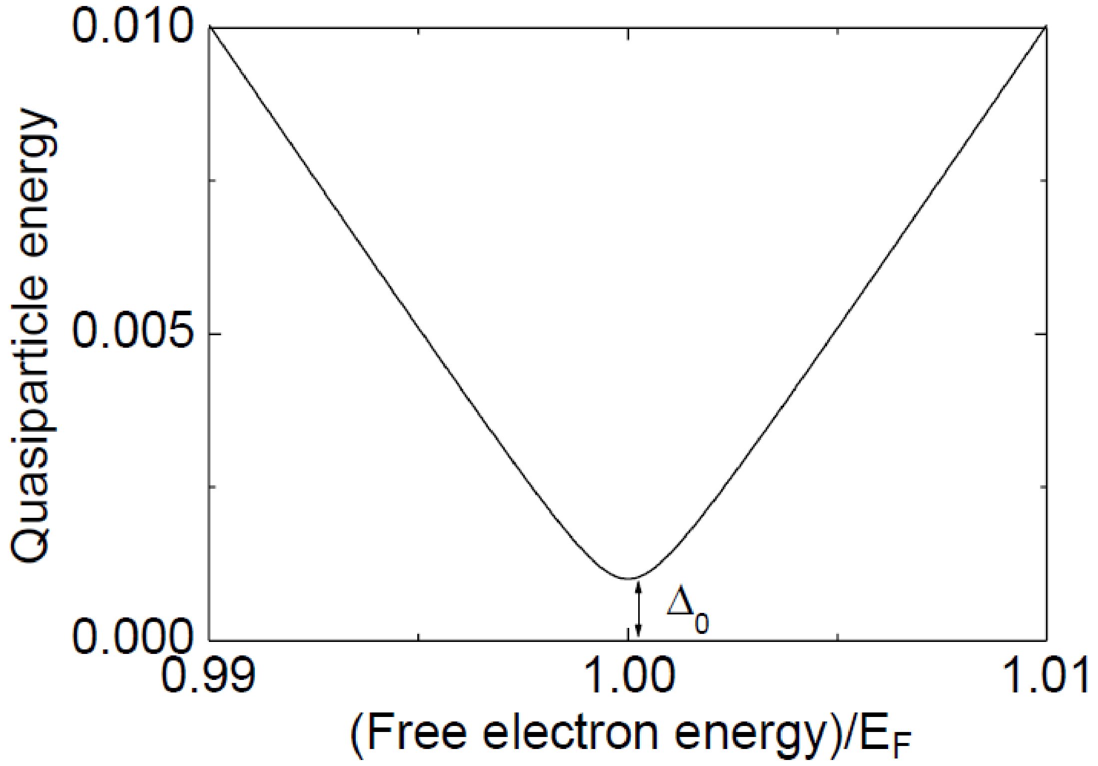
Energy of the unpaired electrons



$$E_k = \sqrt{(\varepsilon_k - E_F)^2 + \Delta_0^2}$$

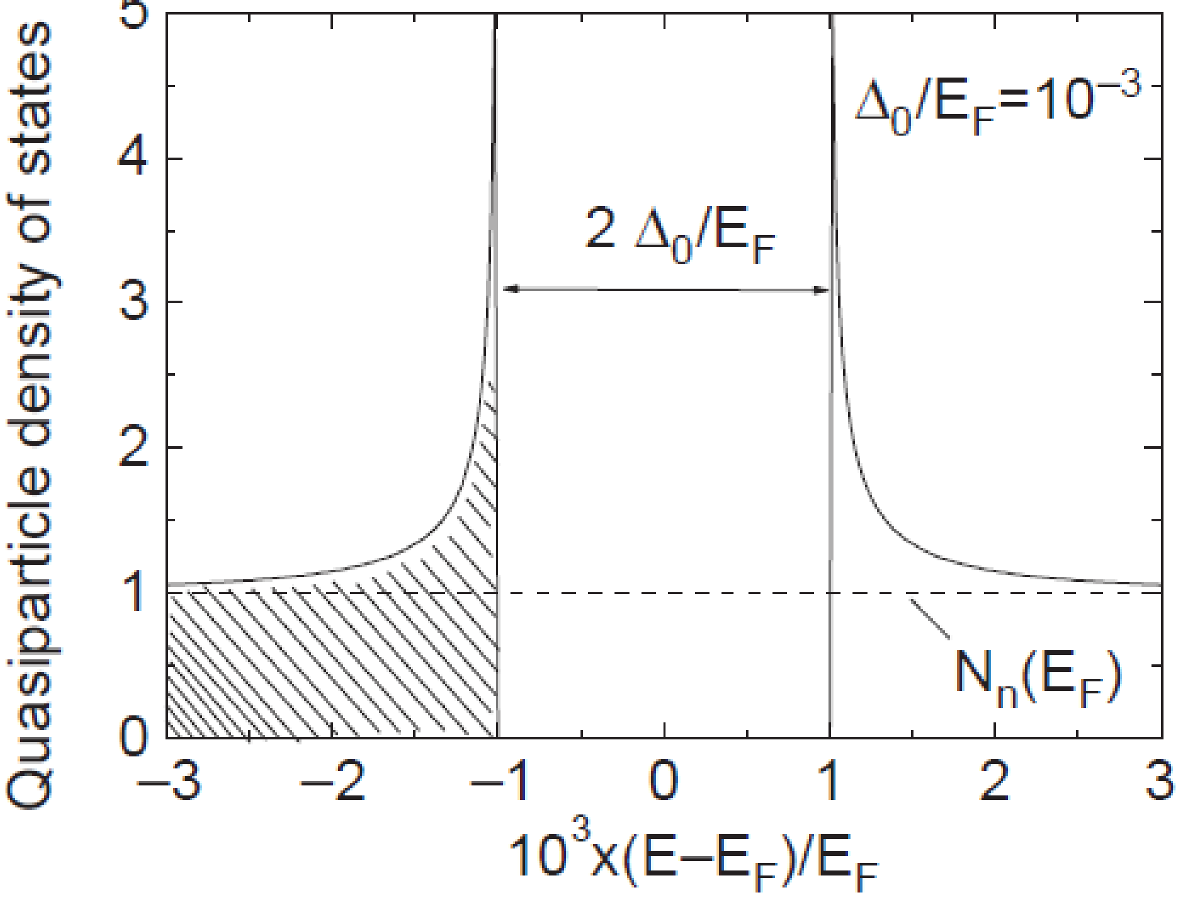
ε_k energy of the individual electrons in the absence of the interaction V





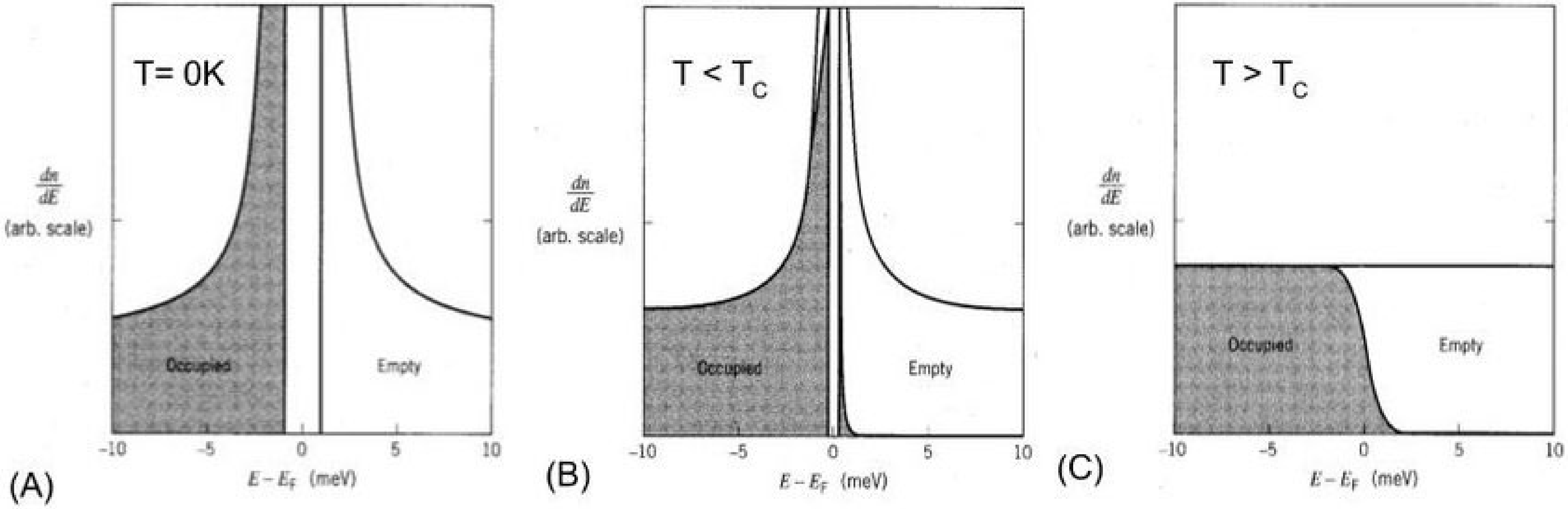
Density of the states

$$N_s(E) = N_n(E_F)\frac{|E - E_F|}{\sqrt{(E - E_F)^2 - \Delta_0^2}}$$



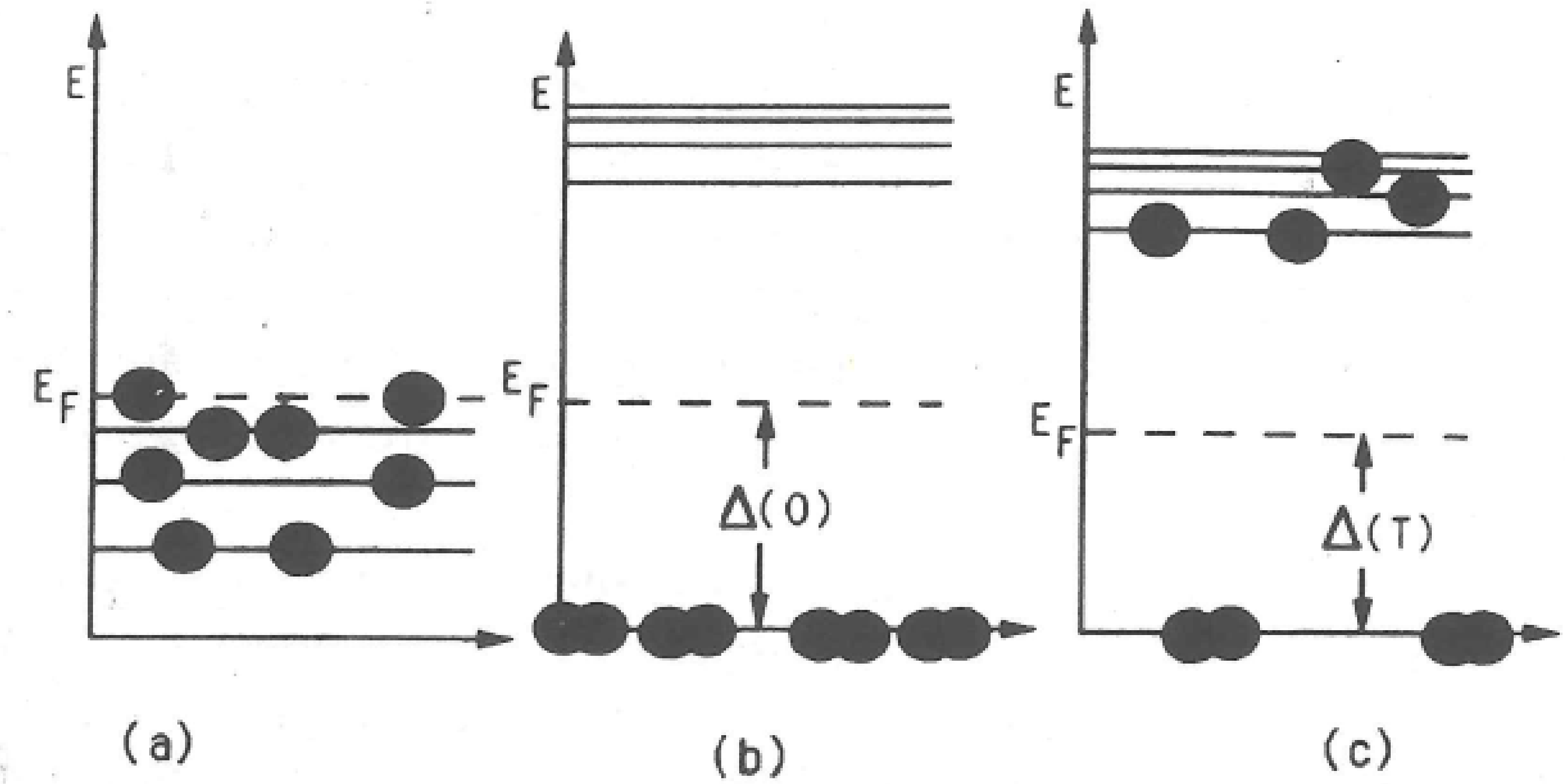
Density of the states (2)

$$N_s(E) = N_n(E_F)\frac{|E - E_F|}{\sqrt{(E - E_F)^2 - \Delta_0^2}}$$



Energy spectrum

Energy spectrum for:
(a) a normal metal. At $T = 0\,K$ all the electrons fill the lower energy levels up to E_F obeying the Pauli exclusion principle.
(b) A superconductor at $T = 0\,K$. The Fermi surface is unstable against pairs formation. The Bose condensation of Cooper pairs into one stable ground state is energetically more stable.
(c) A superconductor at $T \neq 0\,K$. Temperature excitations start to break Cooper pairs and single particle excitations go to fill the energy levels of the "normal bound"

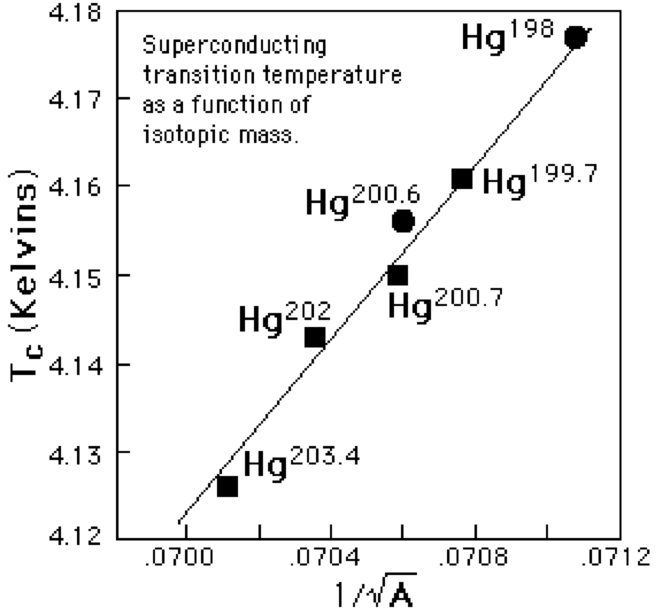


Critical Temperature

$$T_c = 1.13\,\frac{\hbar\omega_D}{K_B}\,e^{\left(-\frac{1}{N_n(E_F)V}\right)}$$

$$T_c = 1.13\,\Theta_D\,e^{\left(-\frac{1}{N_n(E_F)V}\right)}$$

$$T_c \propto M_i^{-1/2}$$



$$T_c = 1.13\,\Theta_D\,e^{\left(-\frac{1}{\lambda_{e-ph}}\right)}$$

$$\lambda_{e-ph} = N_n(E_F)V$$

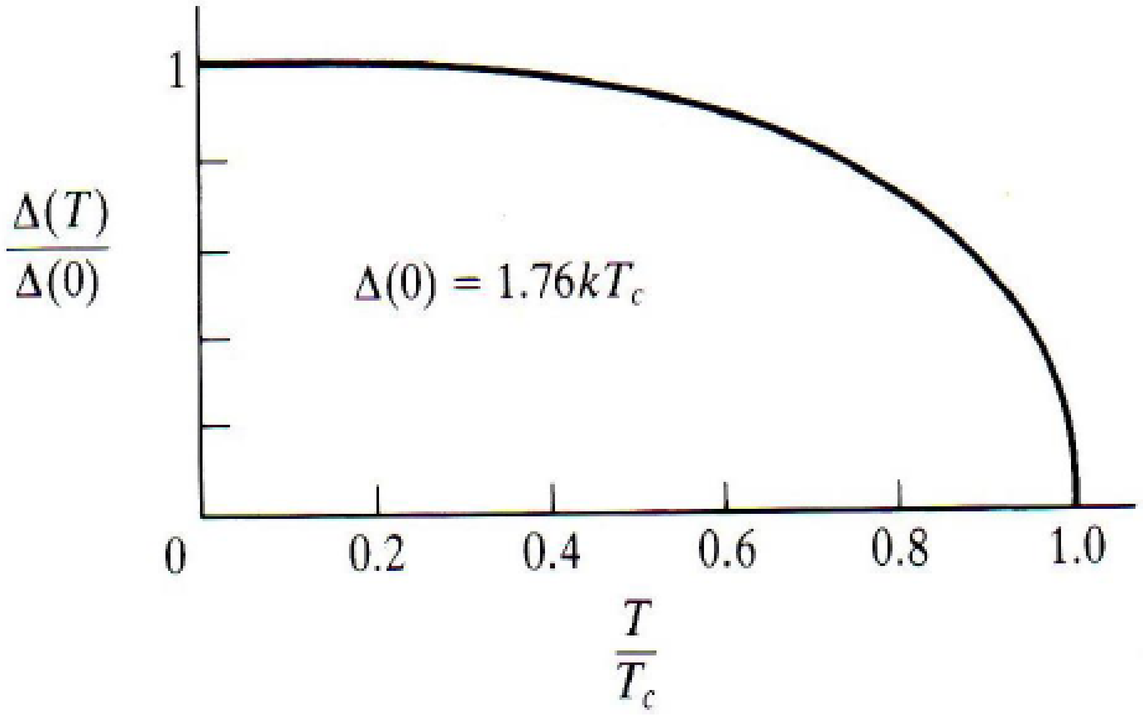
Electron-phonon coupling constant

Energy gap

$$\Delta(0) = 2\hbar\omega_D e^{\left(-\frac{1}{VN_n(E_F)}\right)}$$

$$2\Delta_0 = 3.52K_BT_c$$

$$\frac{\Delta(T)}{\Delta_0} = \sqrt{\cos\left[\frac{\pi}{2}\left(\frac{T}{T_c}\right)^2\right]}$$



Energy gap

$$\Delta(0) = 2\hbar\omega_D e^{\left(-\frac{1}{VN_n(E_F)}\right)}$$

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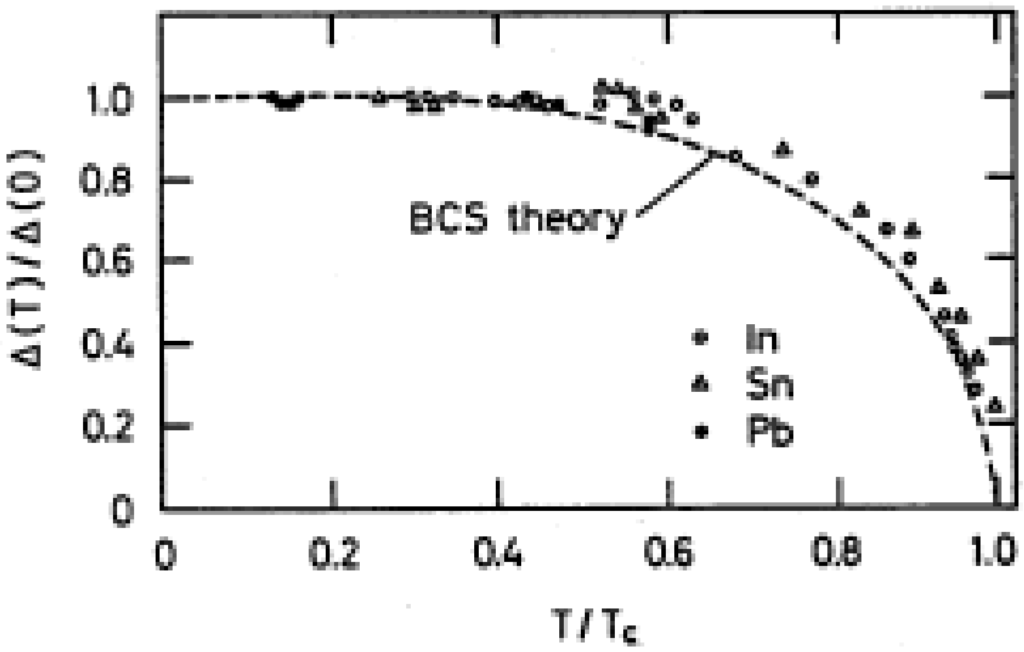


Energy gap additional aspects

1. The energy gap **can have different magnitudes along different crystallographic directions**. Such superconductors are referred to as anisotropic

2. In **some compounds two energy gaps exist (e.g. MgB_2)**. In these materials two or more different Fermi surfaces or energy bands contribute to the superconductivity, each forming its own energy gap, respectively (“two-band superconductivity”)

3. In principle, **the energy gap can be changed even by very small concentrations of impurities**, having an atomic angular momentum and, hence, a magnetic moment (paramagnetic impurities). In this way superconductors can be generated without an energy gap (“gapless superconductor”). However, they are still superconducting, since they display pair correlation

Specific Heath

Normal state (metal)

$$C_{en} = \gamma T$$

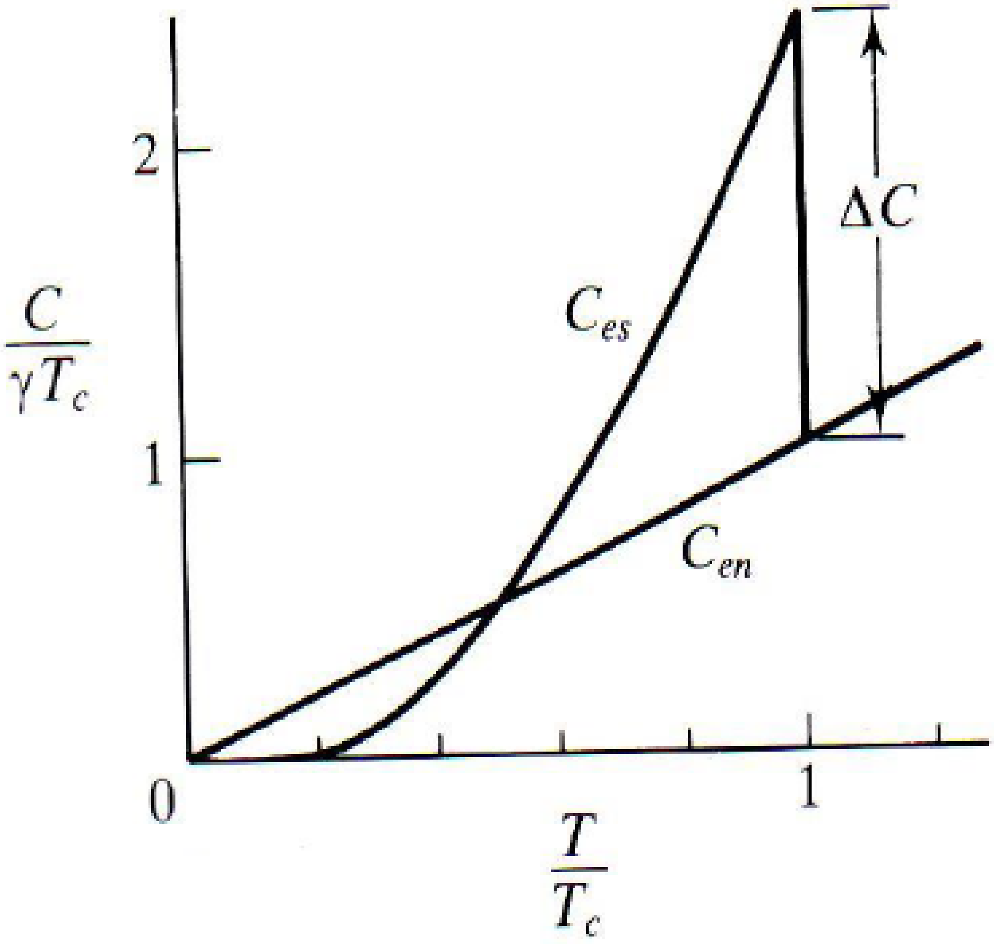
$$\gamma = \frac{2}{3}\pi^2 {k_B}^2 N(E_F)$$

Sommerfeld constant

Superconducting state

$$C_{es} \propto e^{-\frac{\Delta(0)}{K_B T}}$$

$$\Delta C = 1.43 C_{en}$$



Correction to BCS

$\Delta_0 = 1.76 K_B T_c$

Wrong prediction for some SC

element	Sn	In	Tl	Ta	Nb	Hg	Pb
$\Delta(0)/k_B T_c$	1.75	1.8	1.8	1.75	1.75	2.3	2.15

Elhiasberg “Strong Coupling” theory: (1965)

Includes details of the e-ph interaction

Includes the e – e Coulomb repulsion

Elhiasberg “Strong Coupling” theory

BCS

$\lambda_{e\text{-}ph} < 1$

$$T_C = 1.13 \frac{\hbar \omega_D}{k_B} \exp\left[-\frac{1}{\lambda_{e-ph}} \right] \qquad \lambda_{e-ph} = N(E_F)V$$

$$\frac{2\Delta_0}{k_B T_C} = 3.52 \quad ; \quad N_S(E) = \frac{N_n(E)|E|}{\sqrt{E^2 - \Delta^2}}$$

STRONG COUPLING

$\alpha^2 F(\omega)$ spectral density electron-phonon interaction

$$\begin{cases} \lambda_{e-ph} = 2\int_0^\infty \frac{\alpha^2(\omega) F(\omega)}{\omega} d\omega \\ \mu^* = \frac{\mu}{1 + \mu \ln \frac{\Omega_p}{\omega_C}} \\ \langle \omega \rangle = \frac{2}{\lambda} \int_0^\infty \alpha^2 F(\omega) d\omega \end{cases}$$

$$T_C = \frac{\hbar \langle \omega \rangle}{1.2 k_B} \exp\left[-\frac{1.04(1 + \lambda_{e-ph})}{\lambda_{e-ph} - \mu^*(1 + 0.62\lambda_{e-ph})} \right]$$

μ^* e–e screened Coulomb repulsion (~0.1-0.3)

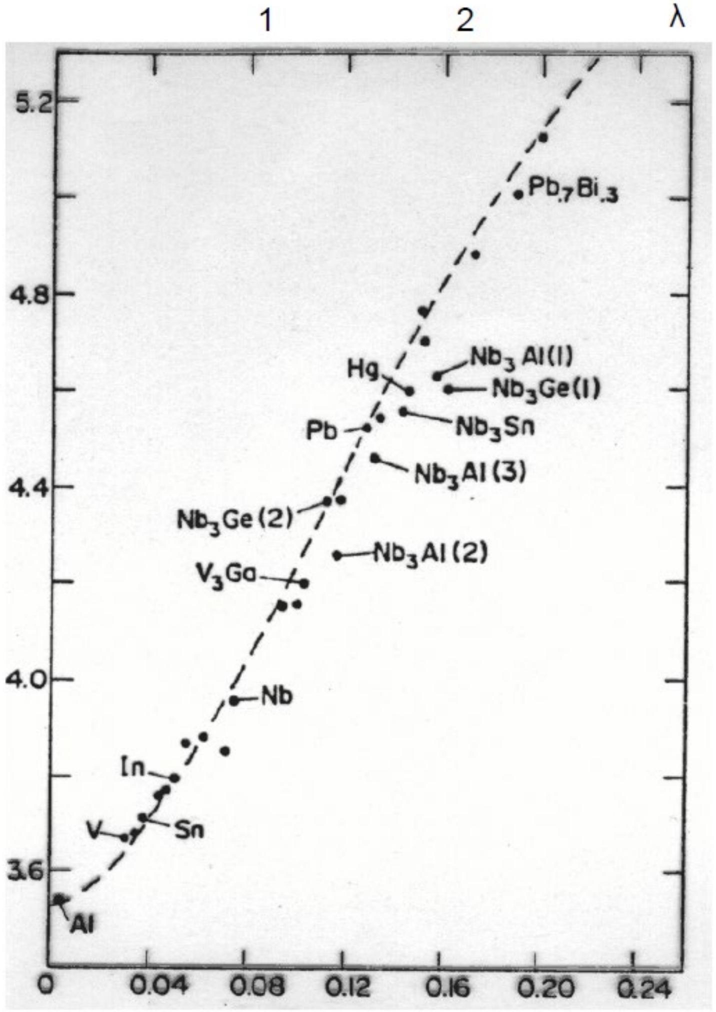
(MacMillan empirical formula)

$$\frac{2\Delta_0}{k_B T_C} = 3.52 \left[1 + A \left(\frac{k_B T_C}{\hbar \omega_D} \right)^2 \ln \left(\frac{\hbar \omega_D}{2 k_B T_C} \right) \right] \qquad A \approx 14$$

$$\begin{cases} N_S(E) = N_n(E) \mathrm{Re} \left[\frac{E}{\sqrt{E^2 - \Delta^2(E)}} \right] \\ \Delta(E) = G(\alpha^2 F(\omega)) \end{cases}$$

$$\alpha = \frac{1}{2} \left[1 - \frac{(1+\lambda)(1+0.62\lambda)\mu^{*2}}{[\lambda - \mu^*(1+0.62\lambda)]^2} \right]$$

Elhiasberg “Strong Coupling” theory



$$\frac{2\Delta_0}{k_B T_C}$$

$$k_B T_C / \hbar\omega_D$$

$$\frac{2\Delta_0}{K_B T_c} = 3.52\eta$$

$$\eta = 1 + A\left(\frac{K_B T_c}{\hbar\omega_D}\right)^2 \ln = 3.52\eta$$

Supercurrent

We apply an electric field $\mathbf{E_0}$ for a short time $\boldsymbol{\delta t}$

Both electrons of a Cooper pair receive an additional momentum:

$$\delta \boldsymbol{p} = -e\boldsymbol{E_0}\delta t$$

all Cooper pairs will have the same non-vanishing momentum:

$$P = \hbar K$$

And the states of the Cooper pairs became:

$$|\boldsymbol{k}\uparrow, -\boldsymbol{k}\downarrow\rangle \Rightarrow \left|\boldsymbol{k} + \frac{\boldsymbol{K}}{\boldsymbol{2}}\uparrow, -\boldsymbol{k} + \frac{\boldsymbol{K}}{\boldsymbol{2}}\downarrow\right\rangle$$

Supercurrent

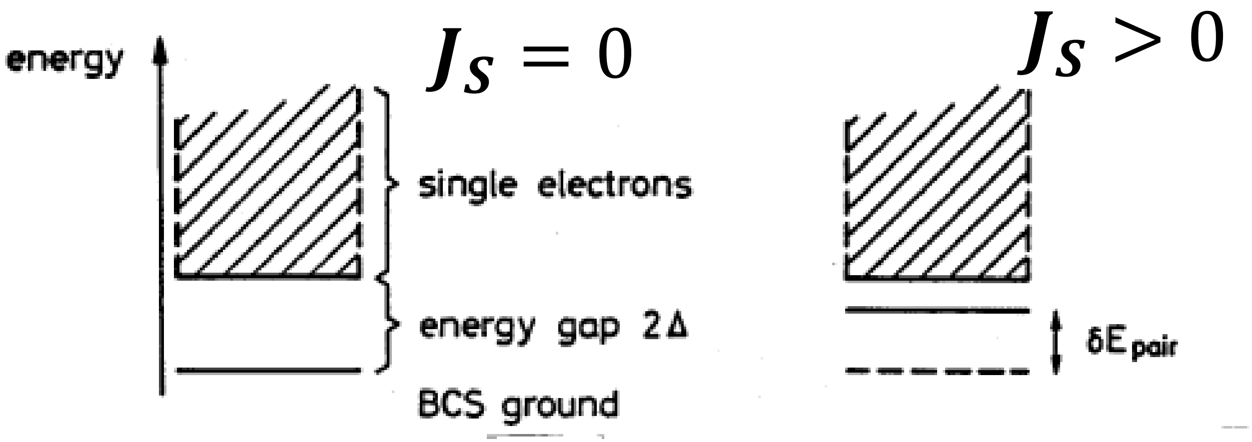
$$\boldsymbol{j_s} = -n_c e \boldsymbol{v}$$

$$m\boldsymbol{v} = \hbar \mathbf{k}$$

It can be shown that the Cooper-pair wave function with a current flowing is simply obtained by multiplying the wave function at rest with the phase factor *exp(i**K**·**R**)* where ***R** = (**r**$_1$ + **r**$_2$)/2* is the coordinate of the center of gravity of the two electrons

$$J_s = -n_c \frac{e\hbar}{m_e} K$$ ➡ **(k, -k) →** $= (\mathbf{k} + \mathbf{K}/2\,, -\mathbf{k} + \mathbf{K}/2)$

All equations of the BCS theory remain applicable and there will remain an energy gap provided the **kinetic-energy gain δE_{pair} of the Cooper pair is less than 2Δ**



Critical current density J_c

To have resistance free-current transport: $2\Delta - \delta E_{pair} > 0$ and $T < T_c(J_S)$

For a single electron the E_k variation is:

$$\delta E_{electron} = \frac{\hbar^2(\mathbf{k}+\mathbf{K}/2)^2}{2m_e} - \frac{\hbar^2 k^2}{2m_e} = \frac{\hbar^2 \mathrm{K}^2}{8m_e} + \frac{\hbar^2 \mathbf{k}\cdot\mathbf{K}}{2m_e} \Rightarrow \delta E_{pair} \approx p_F P / m_e$$

$\delta E_{pair} < 2\Delta$

$J_S = -n_c \frac{e\hbar}{m_e} K$

$$J_s \leq J_c \approx \frac{2en_c\Delta}{p_F}$$

in a long wire

$$H_c = \lambda_L J_c \approx \lambda_L \frac{2en_c\Delta}{p_F}$$

Fundamental parameters in SC

$$\lambda_L = \sqrt{\frac{m_s}{\mu_0 n_s q_s^2}}$$

London Penetration Depth λ_L

$$\xi_{GL} = \sqrt{-\frac{\hbar^2}{2m\alpha}}$$

Ginzburg Landau Coherence lenght ξ_{GL}

$$\kappa = \frac{\lambda_L}{\xi_{GL}}$$

Ginzburg Landau Parameter

$$\xi_{BCS} = 0.18\frac{\hbar v_F}{\pi\Delta}$$

BCS coherence lenght

$$2\Delta_0 = 3.52 K_B T_c$$

Energy gap

superconductivity flash mob video

https://www.youtube.com/watch?v=O6sukIsOozk

Bibliography of this part

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  3.1 Conventional Superconductivity

- Peter Schmuser, Superconductivity

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  https://cds.cern.ch/record/503603

Topics for students' lectures

- Superconducting Magnetic Coils
- Superconducting Cables and Tapes
- Superconducting Permanent Magnets
- Nuclear Magnetic Resonance
- Magnetic Resonance Imaging
- Particle Accelerators
- Nuclear Fusion
- Energy Storage Devices
- Motors and Generators
- Magnetic Separation
- Levitated Trains
- Superconductors for Power Transmission: Cables, Transformers and Current-Limiting Devices
- Resonators for Particle Accelerators
- Resonators and Filters for Communications Technology Superconducting Detectors
- Superconducting Quantum Interference Devices (SQUID)
- Superconductors in Microelectronics
- Voltage Standards
- Digital Electronics Based on Josephson Junction
- Quantum computing