12.06 : riassunto

- dinamica dei sistemi a tempo continuo

$\longrightarrow$ dominio del tempo

$$\dot{x}(t) = F x(t) + G u(t)$$
$$y(t) = H x(t) + J u(t)$$

$$x(t) = x_e(t) + x_f(t)$$
$$= e^{Ft} x_0 + \int_0^t e^{F(t-\tau)} G u(\tau) \, d\tau$$

$$y(t) = y_e(t) + y_f(t)$$
$$= He^{Ft} x_0 + \int_0^t \underbrace{\left( H e^{F(t-\tau)} G + J \delta(t-\tau) \right)}_{W(t-\tau)} u(\tau) \, d\tau$$

- EVOLUZIONE nel DOMINIO della FREQUENZA $(s \in \mathbb{C})$

trasformate di laplace $\quad \mathcal{L}(v(t)) = V(s) = \int_0^{\infty} v(t)\, e^{-st}\, dt$

$$\begin{cases} \dot{x}(t) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + Ju(t) \end{cases}$$

$$\xrightarrow{\mathcal{L}} \begin{cases} \mathcal{L}(\dot{x}(t)) = F \cdot \mathcal{L}(x(t)) + G\,\mathcal{L}(u(t)) \\ \mathcal{L}(y(t)) = H \cdot \mathcal{L}(x(t)) + J\,\mathcal{L}(u(t)) \end{cases}$$

$$\mathcal{L}(x(t)) = X(s) \in \mathbb{R}^n$$
$$\mathcal{L}(u(t)) = U(s) \in \mathbb{R}^m$$
$$\mathcal{L}(y(t)) = Y(s) \in \mathbb{R}^p$$

$$\begin{cases} sX(s) - x_0 = F \cdot X(s) + G\,U(s) \\ Y(s) = H\,X(s) + J \cdot U(s) \end{cases}$$

$$\begin{cases} (sI - F)\,X(s) = x_0 + GU(s) \\ Y(s) = HX(s) + J\,U(s) \end{cases}$$

$$X(s) = \underbrace{(sI-F)^{-1} x_0}_{\mathcal{L}(x_e(t)) = X_e(s)} + \underbrace{(sI-F)^{-1} G\, U(s)}_{X_F(s) = \mathcal{L}(x_f(s))}$$

$$\begin{aligned} Y(s) &= H\,X(s) + J\,U(s) \\ &= H(sI-F)^{-1} x_0 + H(sI-F)^{-1} G\,U(s) + J\,U(s) \\ &= \underbrace{H(sI-F)^{-1} x_0}_{\mathcal{L}(y_e(t)) = Y_e(s)} + \underbrace{\left(H(sI-F)^{-1}G + J\right) U(s)}_{Y_F(s) = \mathcal{L}(y_f(t))} \end{aligned}$$

$$\begin{aligned} Y_F(s) = \mathcal{L}(y_f(t)) &= \mathcal{L}\left(\int w(t-\tau)\, u(\tau)\, d\tau\right) \\ &= \underbrace{\boxed{\left(H(sI-F)^{-1}G + J\right)}}_{\mathbb{R}^{p\times m} \ni \boxed{W(s) = \mathcal{L}(w(t))}} \cdot U(s) \end{aligned}$$

matrice di trasferimento $\quad : \quad W(s) = \dfrac{Y_F(s)}{U(s)}$

comportamento I/O

$$\begin{aligned} X_e(s) = \mathcal{L}(x_e(t)) &= \mathcal{L}(e^{Ft} x_0) \\ &= (sI-F)^{-1} \cdot x_0 \end{aligned} \longrightarrow \mathcal{L}(e^{Ft}) = (sI-F)^{-1}$$

$$\Sigma : \quad \begin{aligned} \dot{x}(t) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) + Ju(t) \end{aligned} \qquad x(0) = x_0$$

$T \in \mathbb{R}^{n \times n}$ : cambio di base $\longrightarrow$ $z(t) = T^{-1} x(t)$

$x(t) = T\, z(t)$

$$\begin{cases} T\dot{z}(t) = FTz(t) + Gu(t) \\ y(t) = HTz(t) + Ju(t) \end{cases}$$

$$\begin{cases} \dot{z}(t) = T^{-1}FT\, z(t) + T^{-1}G\, u(t) \\ y(t) = HT\, z(t) + Ju(t) \end{cases}$$

$$\Sigma' \quad \begin{cases} \dot{z}(t) = F' z(t) + G' u(t) \\ y(t) = H' z(t) + J u(t) \end{cases}$$

$$\begin{aligned} F' &= T^{-1}FT && \in \mathbb{R}^{n \times n} \\ G' &= T^{-1}G && \in \mathbb{R}^{n \times m} \\ H' &= HT && \in \mathbb{R}^{p \times n} \\ J' &= J && \in \mathbb{R}^{p \times m} \end{aligned}$$

$$\Sigma = (F, G, H, J) \xrightarrow{\quad T \quad} \Sigma' = (F', G', H', J')$$

$\Sigma$ e $\Sigma'$ sono algebricamente equivalenti ovvero hanno lo stesso comportamento I/O

$$\begin{aligned} W'(s) &= H'(sI - F')^{-1}G' + J' \\ &= HT(sI - T^{-1}FT)^{-1}\, T^{-1}G + J \\ &= HT\left(T^{-1}(sI - F)T\right)^{-1} T^{-1}G + J \\ &= HTT^{-1}(sI - F)^{-1}\, TT^{-1}G + J \\ &= H(sI - F)^{-1}G + J \\ &= W(s) \end{aligned}$$

$$W(s) = H(sI - F)^{-1}G + J = H \cdot \frac{\operatorname{adj}(sI - F)}{\det(sI - F)}\, G + J$$

$\operatorname{adj}(sI - F)$ : elementi sono polinomi di grado al massimo $n$

$\det(sI - F)$ : polinomio caratteristico di $F$ di grado $n$

$$= \begin{bmatrix} W_{11}(s) & \cdots & W_{1m}(s) \\ \vdots & & \vdots \\ W_{p1}(s) & \cdots & W_{pm}(s) \end{bmatrix} \in \mathbb{R}^{p \times m}$$

$$W_{ij}(s) = \frac{N_{ij}(s)}{D_{ij}(s)}$$

con $\deg D_{ij}(s) \geq \deg N_{ij}(s)$

$\longrightarrow$ se $p \in \mathbb{C}$ è un polo di $W(s)$ allora $p \in \mathbb{C}$ è un polo di almeno un $W_{ij}(s)$

$\{ \text{poli } W(s) \} \subseteq \{ \text{autovalori } F \}$

# DINAMICA dei SISTEMI a TEMPO DISCRETO

$\Sigma$ LTI

$$x(t+1) = Fx(t) + Gu(t) \qquad t \in \mathbb{N} \qquad F \in \mathbb{R}^{n \times n},\ G \in \mathbb{R}^{n \times m}$$
$$y(t) = Hx(t) + Ju(t) \qquad H \in \mathbb{R}^{p \times n},\ J \in \mathbb{R}^{p \times m}$$

- EVOLUZIONE nel DOMINIO del TEMPO

sistema autonomo scalare $(x(t) \in \mathbb{R},\ u(t) = 0)$

$$x(t+1) = f\,x(t),\quad x(0) = x_0 \longrightarrow x(t) = f^t \cdot x_0$$

sistema autonomo multidimensionale $(x(t) \in \mathbb{R}^n,\ u(t) = 0)$

$$x(t+1) = F\,x(t),\quad x(0) = x_0 \longrightarrow x(t) = F^t x_0$$

$$
\begin{aligned}
x(0) &= x_0 \\
x(1) &= F \cdot x(0) = F x_0 \\
x(2) &= F x(1) = F \cdot F \cdot x_0 = F^2 x_0 \\
x(3) &= F\, x(2) = F \cdot F^2 x_0 = F^3 x_0 \\
&\vdots \\
x(k) &= F x(k-1) = \dots = F^k x_0
\end{aligned}
$$

come calcolare $F^t$, $F \in \mathbb{R}^{n \times n}$, in "modo furbo"?

- diagonalizzare $F$ : trovare una matrice $T \in \mathbb{R}^{n \times n}$ tale che

$$T^{-1} F T = F_J$$

$$F^t = T \cdot F_J^t \cdot T^{-1}$$

$$F_J = \begin{bmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_k} \end{bmatrix} \in \mathbb{R}^{n \times n} \longrightarrow F_J^t = \begin{bmatrix} J_{\lambda_1}^t & & \\ & \ddots & \\ & & J_{\lambda_k}^t \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$J_{\lambda_i} = \begin{bmatrix} J_{\lambda_i,1} & & \\ & \ddots & \\ & & J_{\lambda_i,m_i^g} \end{bmatrix} \in \mathbb{R}^{m_i^a \times m_i^a} \longrightarrow J_{\lambda_i}^t = \begin{bmatrix} J_{\lambda_i,1}^t & & \\ & \ddots & \\ & & J_{\lambda_i,m_i^g}^t \end{bmatrix} \in \mathbb{R}^{m_i^a \times m_i^a}$$

$$J_{\lambda_i,j} = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{m^r_{ij} \times m^r_{ij}}$$

(ii) $\lambda_i \neq 0$

$$J_{\lambda_i,j} = \lambda_i I + N \qquad N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in \mathbb{R}^{m^r_{ij} \times m^r_{ij}}$$

matrice nilpotente con indice di nilpotenza pari a $m^r_{ij}$

$$N^0 = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \qquad N^1 = N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 0 \\ & & & & & 0 \end{bmatrix} \quad \dots \quad N^{m^r_{ij}-1} = \begin{bmatrix} 0 & \cdots & 1 \\ & \ddots & \vdots \\ & & 0 \end{bmatrix} \qquad N^{m^r_{ij}} = 0$$

$\lambda_i I, N$ : commutano

Siano $A, B \in \mathbb{R}^{n \times n}$ tali che $AB = BA$
allora
$$(A+B)^t = \sum_{k=0}^{t} \binom{t}{k} A^{t-k} B^k = \sum_{k=0}^{t} \frac{t!}{(t-k)!\,k!} A^{t-k} B^k$$

$$\begin{aligned} J^t_{\lambda_i,i} &= (\lambda_i I + N)^t \\ &= \sum_{k=0}^{t} \binom{t}{k} (\lambda_i I)^{t-k} N^k \\ &= (t \geq m^r_{ij}) = \binom{t}{0} \lambda_i^t + \binom{t}{1} \lambda_i^{t-1} N + \binom{t}{2} \lambda_i^{t-2} N^2 + \dots \\ &\quad + \dots + \binom{t}{m^r_{ij}-1} \lambda_i^{t-(m^r_{ij}-1)} N^{m^r_{ij}-1} \end{aligned}$$

$$J^t_{\lambda_i,i} = \begin{bmatrix} \binom{t}{0}\lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \cdots & \binom{t}{m^r_{ij}-1}\lambda_i^{t-(m^r_{ij}-1)} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \binom{t}{1}\lambda_i^{t-1} \\ & & & \binom{t}{0}\lambda_i^t \end{bmatrix}$$

(ii) $\lambda_i = 0$

$$J_{\lambda_i,j} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} = N$$

$$\delta(t) = \begin{cases} 1 & t=0 \\ 0 & t\neq 0 \end{cases}$$

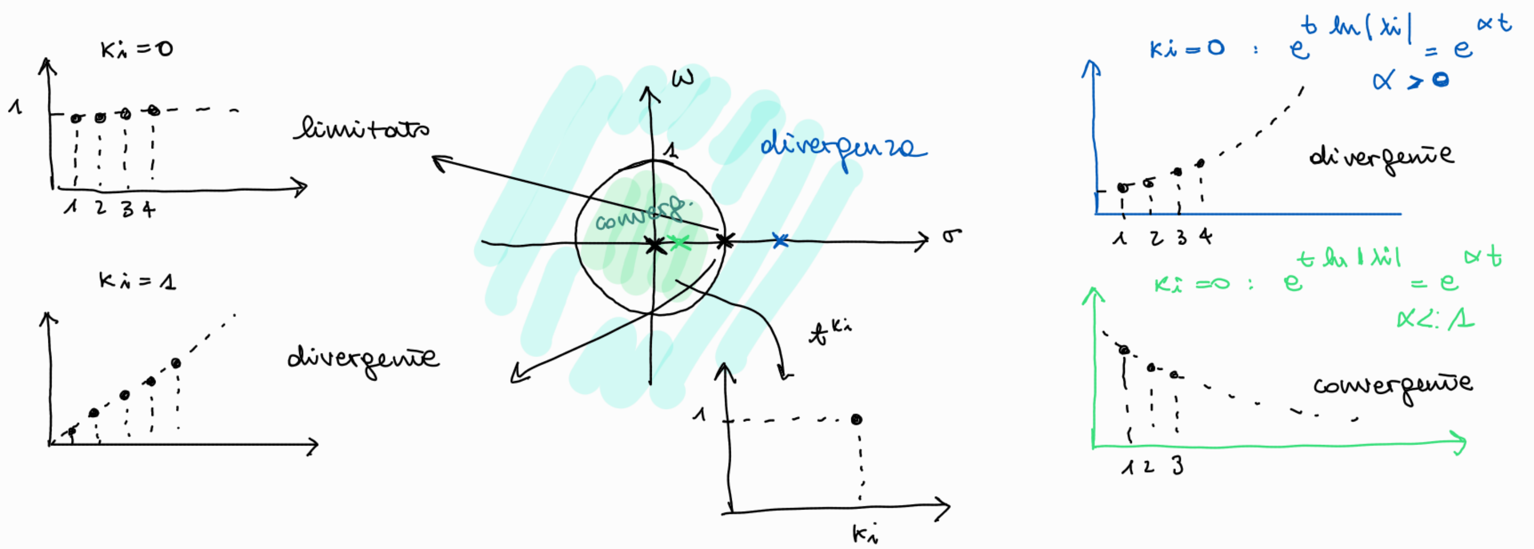
detta di Kronecker (impulso discreto)

$$J^t_{\lambda_i,j} = \begin{bmatrix} \delta(t) & \delta(t-1) & \cdots & \delta(t-(m^i_j-1)) \\ & \ddots & \ddots & \vdots \\ & & & \delta(t-1) \\ & & & \delta(t) \end{bmatrix}$$

$\binom{t}{0}\lambda_i^t$, $\binom{t}{1}\lambda_i^{t-1}$, ... $\binom{t}{m^i_j-1}\lambda_i^{t-(m^i_j-1)}$ modi elementari del sistema

$\delta(t)$, $\delta(t-1)$ ... $\delta(t-(m^i_j-1))$

- se $\lambda_i \neq 0$ allora $\binom{t}{k}\lambda_i^{t-k} \sim t^k \lambda_i^t = t^k e^{\ln(\lambda_i^t)} = t^k \cdot e^{t\cdot\ln(\lambda_i)}$
- se $\lambda_i = 0$ allora i modi elementari sono $\delta(t) \dots \delta(t-(m^i_j-1))$ : si annullano dopo un numero finito di passi
- se $\lambda_i = \sigma_i + i\,\omega_i \in \mathbb{C}$ è un autovalore di $F$ allora
  - $\lambda_i \neq 0$ il modo corrispondente è $\binom{t}{k_i}\lambda_i^{t-k_i} \sim t^{k_i}\cdot e^{t\ln(\lambda_i)} = t^{k_i} e^{t(\ln(|\lambda_i|)+i\arg(\lambda_i))}$
  - $\lambda_i = 0$ il modo corrispondente è $\delta(t-k_i)$



sistema autonomo multidimensionale

$$x(t+1) = F x(t) \quad , \quad x(0) = x_0 \qquad \text{allora} \qquad x(t) = F^t x_0 = x_\ell(t) \qquad x_\ell(t): \text{evoluzione libera}$$

$$y(t) = H x(t) \qquad \text{allora} \qquad y(t) = y_\ell(t) = H F^t x_0$$

= combinazione lineare dei modi elementari del sistema

comportamento asintotico

•) $|\lambda_i| < 1 \quad \forall i \qquad F^t \xrightarrow{t \to \infty} 0 \quad \Rightarrow \quad y(t) = H F^t x_0 \xrightarrow{t \to \infty} 0$

•) $|\lambda_i| \leq 1 \quad \forall i$ e $m_i^a = m_i^g$ se $|\lambda_i| = 1 \qquad F^t$ limitata $\Rightarrow \quad y(t) = H F^t x_0$ limitata $\forall\, H, x_0$

•) $\exists\, \lambda_i$ tale $|\lambda_i| > 1$ o $|\lambda_i| = 1$ e $m_i^a > m_i^g \qquad F^t$ non limitata $\Rightarrow \quad y(t)$ ? dipende da $H$ e $x_0$

sistema non autonomo multidimensionale $(x(t) \in \mathbb{R}^n,\ u(t) \neq 0)$

$$x(t+1) = F x(t) + G u(t) \quad , \quad x(0) = x_0 \qquad \text{allora} \qquad x(t) = x_\ell(t) + x_f(t) \qquad x_f(t): \text{ev. forzata}$$

$$y(t) = H x(t) + J u(t) \qquad \text{allora} \qquad y(t) = y_\ell(t) + y_f(t)$$

$$x(1) = F x(0) + G u(0)$$

$$x(2) = F x(1) + G u(1) \overset{!}{=} F(F x(0) + G u(0)) + G u(1) = F^2 x_0 + F G u(0) + G u(1)$$

$$x(3) = F x(2) + G u(2) \overset{!}{=} F(F^2 x_0 + F G u(0) + G u(1)) + G u(2) = F^3 x_0 + F^2 G u(0) + F G u(1) + G u(2)$$

$\vdots$

$$x(t) = F^t x_0 + \sum_{k=0}^{t-1} F^{t-1-k} G u(k)$$

$$\overset{!}{=} F^t x_0 + \underbrace{\begin{bmatrix} G & FG & F^2G & \cdots & F^{t-1}G \end{bmatrix}}_{R_t \,\in\, \mathbb{R}^{n \times mt}} \underbrace{\begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}}_{u_t \,\in\, \mathbb{R}^{mt}} = x_\ell(t) + x_f(t)$$

matrice di raggiungibilità in $t$ passi

$$y(t) = H x(t) + J u(t)$$

$$\overset{!}{=} H F^t x_0 + \sum_{k=0}^{t-1} H F^{t-1-k} G u(k) + J u(t)$$

$$\overset{!}{=} H F^t x_0 + \left( H R_t u_t + J u(t) \right) = y_\ell(t) + y_f(t)$$

$$y_f(t) = \sum_{k=0}^{t-1} HF^{t-1-k} G u(k) + J u(t) = \sum_{-\infty}^{+\infty} w(t-k)\, u(k)$$

$$w(t) = \begin{cases} J & t = 0 \\ HF^{t-1} G & t \neq 0 \end{cases}$$

risposta impulsiva del sistema