② DINAMICA di SISTEMI DINAMICI

$\underbrace{\text{DINAMICA}}$ └ evoluzione temporale del sistema
delle variabili di interesse (stato | uscita)

DINAMICA di SISTEMI a TEMPO CONTINUO

Σ LTI

$$\dot{x}(t) = F x(t) + G u(t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ y \in \mathbb{R}^p$$
$$y(t) = H x(t) + J u(t) \qquad F \in \mathbb{R}^{n\times n},\ G \in \mathbb{R}^{n \times m}$$
$$H \in \mathbb{R}^{p\times n},\ J \in \mathbb{R}^{p\times m}$$

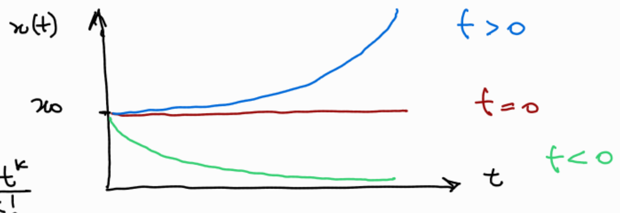
- EVOLUZIONE nel DOMINIO del TEMPO $(t \in \mathbb{R}^+)$

- sistema autonomo scalare $(x \in \mathbb{R},\ u = 0)$

$$\Sigma : \quad \dot{x}(t) = f x(t), \quad f \in \mathbb{R}, \quad x(0) = x_0 \longrightarrow \boxed{x(t) = e^{ft} x_0}$$

si verifica che

$$\dot{x}(t) = f \cdot e^{ft} x_0 = f x(t)$$



$$\triangleright\ e^{ft} = \left(1 + f \cdot t + \frac{f^2 \cdot t^2}{2!} + \dots\right) = \sum_{k=0}^{\infty} \frac{f^k t^k}{k!}$$

- sistema autonomo multivariato $(x \in \mathbb{R}^n,\ u = 0)$

$$\Sigma : \quad \dot{x}(t) = F x(t), \quad F \in \mathbb{R}^{n\times n}, \quad x(0) = x_0 \longrightarrow \boxed{x(t) = e^{Ft} x_0}$$

si verifica che

$$\dot{x}(t) = F \cdot e^{Ft} x_0 = F \cdot x(t)$$

$$\triangleright\ e^{Ft} = \left(I + Ft + \frac{F^2 t^2}{2!} + \dots\right) = \sum_{k=0}^{\infty} \frac{F^k t^k}{k!}$$

come calcolare e^{Ft}, $F \in \mathbb{R}^{n\times n}$, in "modo furbo"?

① calcolo diretto : $e^{Ft} = \sum_{k=0}^{\infty} \frac{F^k t^k}{k!}$
→ casi semplici strutturati

esempi

1. $F = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \longrightarrow e^{Ft} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$

$$e^{Ft} = \sum_{k=0}^{\infty} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^k \cdot \frac{t^k}{k!} = \sum_{k=0}^{\infty} \begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix} \frac{t^k}{k!}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} 1^k \cdot \frac{t^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} 2^k \frac{t^k}{k!} \end{bmatrix} = \begin{bmatrix} e^{1 \cdot t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

in generale per matrici diagonali

$$F = \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_n \end{bmatrix} \longrightarrow e^{Ft} = \begin{bmatrix} e^{f_1 t} & & \\ & \ddots & \\ & & e^{f_n t} \end{bmatrix}$$

2. $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = I + N$ con $NI = IN$

N : matrice nilpotente

$\exists \bar{K} \in \mathbb{N}$ tale che $N^K = 0 \quad \forall K \geq \bar{K}$

indice di nilpotenza

$$N^0 = I \qquad N^1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = N^3 = N^4 = \dots$$

$$e^{Nt} = \sum_{K=0}^{\infty} \frac{N^K t^K}{K!} = \sum_{K=0}^{\bar{K}=2} \frac{N^K t^K}{K!} = N^0 + Nt + \frac{N^2 t^2}{2} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$e^{Ft} = e^{(N+I)t} = e^{It} \cdot e^{Nt} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & t \cdot e^t \\ 0 & e^t \end{bmatrix}$$

3. $F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = I + N$

N : matrice nilpotente

$$N^0 = I \qquad N^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{Nt} = \sum_{K=0}^{\infty} \frac{N^K t^K}{K!} = \sum_{K=0}^{\bar{K}=3} \frac{N^K t^K}{K!} = N^0 + N^1 t + \frac{N^2 t^2}{2} + \frac{N^3 t^3}{3!} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{Ft} = e^{(I+N)t} = e^{It} \cdot e^{Nt} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & te^t & \frac{t^2}{2} e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}$$

in generale per matrici quasi-diagonali

$$F = \begin{bmatrix} f & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & f \end{bmatrix} \longrightarrow e^{Ft} = \begin{bmatrix} e^{ft} & te^{ft} & \dots & \frac{t^{n-1}}{(n-1)!} e^{ft} \\ & \ddots & \ddots & \vdots \\ & & \ddots & te^{ft} \\ & & & e^{ft} \end{bmatrix}$$

- diagonalizzazione di F $\rightarrow$ casi più complessi : trovare una matrice $T \in \mathbb{R}^{n \times n}$ tale

$$T^{-1} F T = F_J$$

$$F_J = \begin{bmatrix} \boxed{J_{\lambda_1}} & & \\ & \ddots & \\ & & \boxed{J_{\lambda_k}} \end{bmatrix} \in \mathbb{R}^{n \times n} \longrightarrow e^{F_J t} = \begin{bmatrix} \boxed{e^{J_{\lambda_1} t}} & & \\ & \ddots & \\ & & \boxed{e^{J_{\lambda_k} t}} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$\{\lambda_1 \dots \lambda_k\} = \Lambda(F)$

$$J_{\lambda_i} = \begin{bmatrix} \boxed{J_{\lambda_i,1}} & & \\ & \ddots & \\ & & \boxed{J_{\lambda_i,m_i^g}} \end{bmatrix} \in \mathbb{R}^{m_i^a \times m_i^a} \longrightarrow e^{J_{\lambda_i} t} = \begin{bmatrix} \boxed{e^{J_{\lambda_i,1} t}} & & \\ & \ddots & \\ & & \boxed{e^{J_{\lambda_i,m_i^g}}} \end{bmatrix} \in \mathbb{R}^{m_i^a \times m_i^a}$$

$$J_{\lambda_i,j} = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{R}^{m_{ij}^r \times m_{ij}^r} \longrightarrow e^{J_{\lambda_i,j} t} = \begin{bmatrix} e^{\lambda_i t} & t\cdot e^{\lambda_i t} & \cdots & \frac{t^{m_{ij}^r-1}}{(m_{ij}^r-1)!} e^{\lambda_i t} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t\cdot e^{\lambda_i t} \\ & & & e^{\lambda_i t} \end{bmatrix} \in \mathbb{R}^{m_{ij}^r \times m_{ij}^r}$$

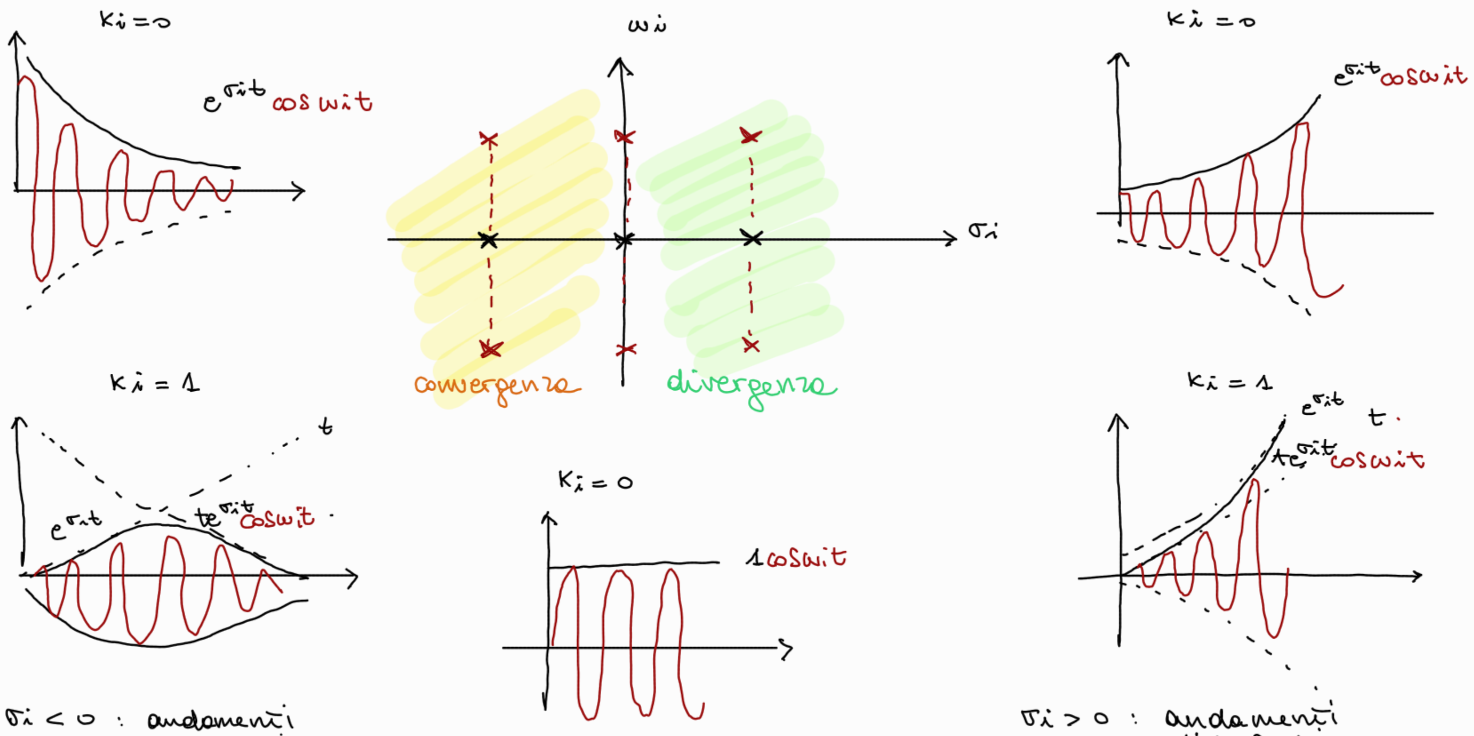
$e^{\lambda_i t}$, $t\cdot e^{\lambda_i t}$, $\frac{t^2}{2} e^{\lambda_i t}$, $\dots$, $\frac{t^{m_{ij}^r-1}}{(m_{ij}^r-1)!} e^{\lambda_i t}$: modi elementari del sistema

- il numero di modi distinti associati a un certo autovalore λ_i è uguale alla dimensione del più grande miniblocco di J_{λ_i}
- se F è diagonalizzabile allora i modi elementari sono esponenziali puri $(e^{\lambda_i t})$
- se $\lambda = \sigma + i\omega \in \mathbb{C}$ è un autovalore di F allora anche $\bar{\lambda} = \sigma - i\omega \in \mathbb{C}$ è un autovalore di F

  i modi associati a $\lambda \in \mathbb{C}$ sono reali $(t^k e^{\sigma t}\cos(\omega t),\ t^k e^{\sigma t}\sin(\omega t))$

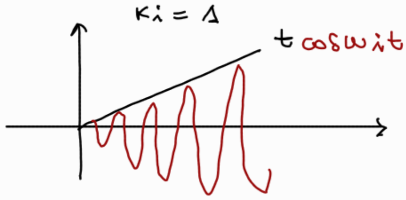
$$\left[e^{Ft}\right]_{hk} = c\cdot e^{\lambda t} + \bar{c}\, e^{\bar{\lambda} t} = (a+ib)e^{(\sigma+i\omega)t} + (a-ib)\, e^{(\sigma-i\omega)t}$$
$$= (a+ib)\, e^{\sigma t}(\cos\omega t + i\sin\omega t) + (a-ib)\, e^{\sigma t}(\cos\omega t - i\sin\omega t)$$
$$= 2a\, e^{\sigma t}\cos\omega t - 2b\, e^{\sigma t}\sin\omega t$$

$$\lambda_i \in \mathbb{C} \longrightarrow t^{k_i} e^{\lambda_i t} = t^{k_i} e^{\sigma_i t}(\cos\omega_i t + i\sin\omega_i t)$$



$\sigma_i < 0$: andamenti convergenti dei modi

$\sigma_i > 0$: andamenti divergenti dei modi



$\sigma_i = 0$: andamenti limitati o divergenti dei modi

Sistema autonomo multidimensionale $(x(t) \in \mathbb{R}^n , u(t) = 0)$

$$\Sigma : \quad \dot{x}(t) = F x(t) \qquad x(0) = x_0 \qquad x(t) = e^{Ft} \cdot x_0 = x_\ell(t) \qquad x_\ell(t) : \text{evoluzione libera}$$

$$y(t) = H x(t) \qquad y(t) = H \cdot x(t) = H \cdot x_\ell(t) \overset{!}{=} H \cdot e^{Ft} \cdot x_0$$

: combinazione lineari di vettori dei modi elementari

F con autovalori $\{\lambda_i\}_{i=1}^{k}$

- $\mathrm{Re}[\lambda_i] < 0 \quad \forall \lambda_i \quad \longleftrightarrow \quad e^{Ft} \xrightarrow{t \to \infty} 0 \quad \Longrightarrow \quad y(t) = H e^{Ft} x_0 \xrightarrow{t \to \infty} 0$
- $\mathrm{Re}[\lambda_i] \leq 0 \quad \forall \lambda_i$ e $m_i^a = m_i^g$ se $\mathrm{Re}[\lambda_i] = 0 \quad \longleftrightarrow \quad e^{Ft}$ limitata $\Longrightarrow \quad y(t) = H e^{Ft} x_0$ limitata $\forall H, x_0$
- $\exists \lambda_i$ t.c. $\mathrm{Re}[\lambda_i] > 0$ oppure $\mathrm{Re}[\lambda_i] = 0$ e $m_i^a > m_i^g \quad \longleftrightarrow \quad e^{Ft}$ non limitata $\Longrightarrow \quad y(t)$? dipendente da H, x_0

Sistema LTI NON autonomo multidimensionale $(x(t) \in \mathbb{R}^n , u(t) \neq 0)$

$$\Sigma : \quad \dot{x}(t) = F x(t) + G u(t) \qquad x(0) = x_0 \qquad x(t) = x_\ell(t) + x_f(t) \qquad x_f(t) : \text{evoluzione forzata}$$

$$y(t) = H x(t) + J u(t) \qquad y(t) = y_\ell(t) + y_f(t)$$

$$\dot{x}(\tau) = F x(\tau) + G u(\tau) \longleftrightarrow e^{-F\tau} \dot{x}(\tau) = e^{-F\tau} F x(\tau) + e^{-F\tau} G u(\tau)$$

$$\longleftrightarrow e^{-F\tau} \dot{x}(\tau) - e^{-F\tau} F x(\tau) = e^{-F\tau} G u(\tau)$$

$$\longleftrightarrow \frac{d}{d\tau} \left(e^{-F\tau} x(\tau) \right) = e^{-F\tau} G u(\tau)$$

allora

$$\int_0^t \frac{d}{d\tau} \left(e^{-F\tau} x(\tau) \right) d\tau = \int_0^t e^{-F\tau} G u(\tau) \, d\tau \longleftrightarrow e^{-Ft} x(t) - e^{-F \cdot 0} x(0) = \int_0^t e^{-F\tau} G u(\tau) \, d\tau$$

$$\longleftrightarrow e^{-Ft} x(t) - x_0 = \int_0^t e^{-F\tau} G u(\tau) \, d\tau$$

$$\longleftrightarrow e^{-Ft} x(t) = x_0 + \int_0^t e^{-F\tau} G u(\tau) \, d\tau$$

$$\longleftrightarrow e^{Ft} e^{-Ft} x(t) = e^{Ft} x_0 + e^{Ft} \int_0^t e^{-F\tau} G u(\tau) \, d\tau$$

$$x(t) = \boxed{e^{Ft} x_0} + \boxed{\int_0^t e^{F(t-\tau)} G u(\tau) \, d\tau} = x_\ell(t) + x_f(t)$$

Di conseguenza

$$
\begin{aligned}
y(t) &= H x(t) + J u(t) \\
&= H\left(e^{Ft} x_0 + \int_0^t e^{F(t-\tau)} G u(\tau)\, d\tau \right) + J u(t) \\
&= \boxed{H e^{Ft} x_0} + \boxed{\int_0^t \left(H e^{F(t-\tau)} G + J \delta(t-\tau) \right) u(\tau)\, d\tau} \\
&= y_e(t) + y_f(t)
\end{aligned}
$$

$$J u(t) = \int_0^t J \delta(t-\tau) u(\tau)\, d\tau$$

$\delta(\cdot)$ rappresenta la delta di Dirac

$$
\begin{aligned}
y_f(t) &= \int_0^t \left(H e^{F(t-\tau)} G + J \delta(t-\tau) \right) u(\tau)\, d\tau \\
&= \int_{-\infty}^{+\infty} w(t-\tau)\, u(\tau)\, d\tau \\
&= (w * u)(t)
\end{aligned}
$$

$$w(t) = H e^{F(t)} G + J \delta(t)$$

risposta impulsiva del sistema