Superconductive Materials

Part 3

London equations

Overview of principal theories

London Theory (1935)

Based on Maxwell´s equations

Phenomenological theory that can describe zero resistance and the Meissner effect

Applicable even to Type-II superconductors after Pippard corrections

Ginzburg Landau Theory (1950)

Phenomenological theory

Can describe non-local effects

Works well near Tc and for Type-II superconductors

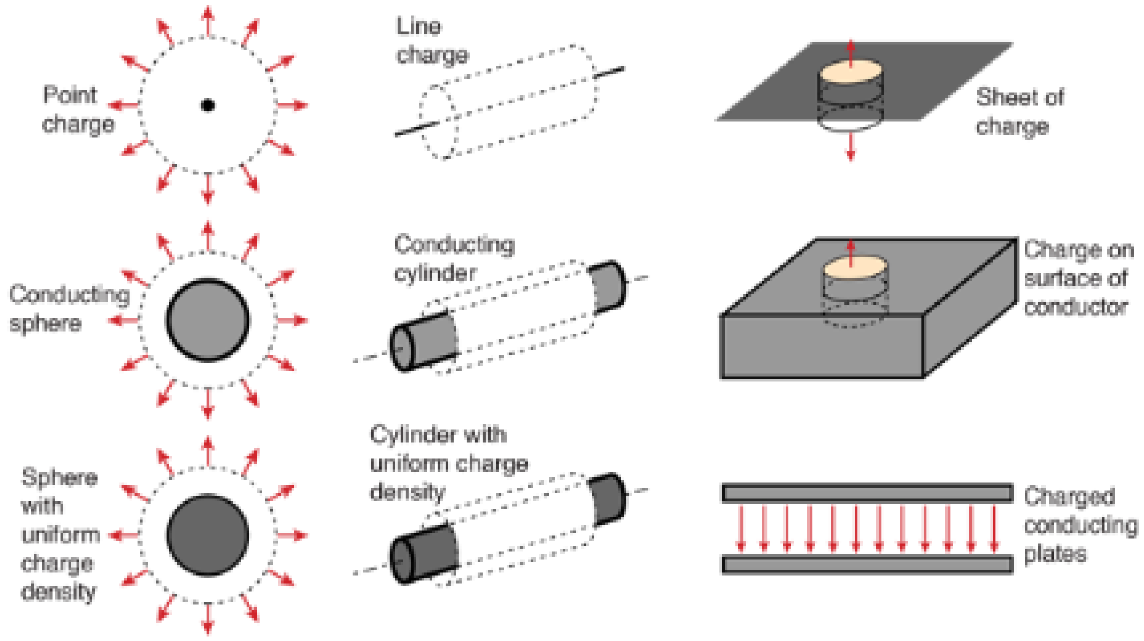
BCS Theory (1957)

First microscopic theory of superconductivity

Published 46 years after the discovery of superconductivity

Maxwell Equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$ **Gauss' Law for electricity**



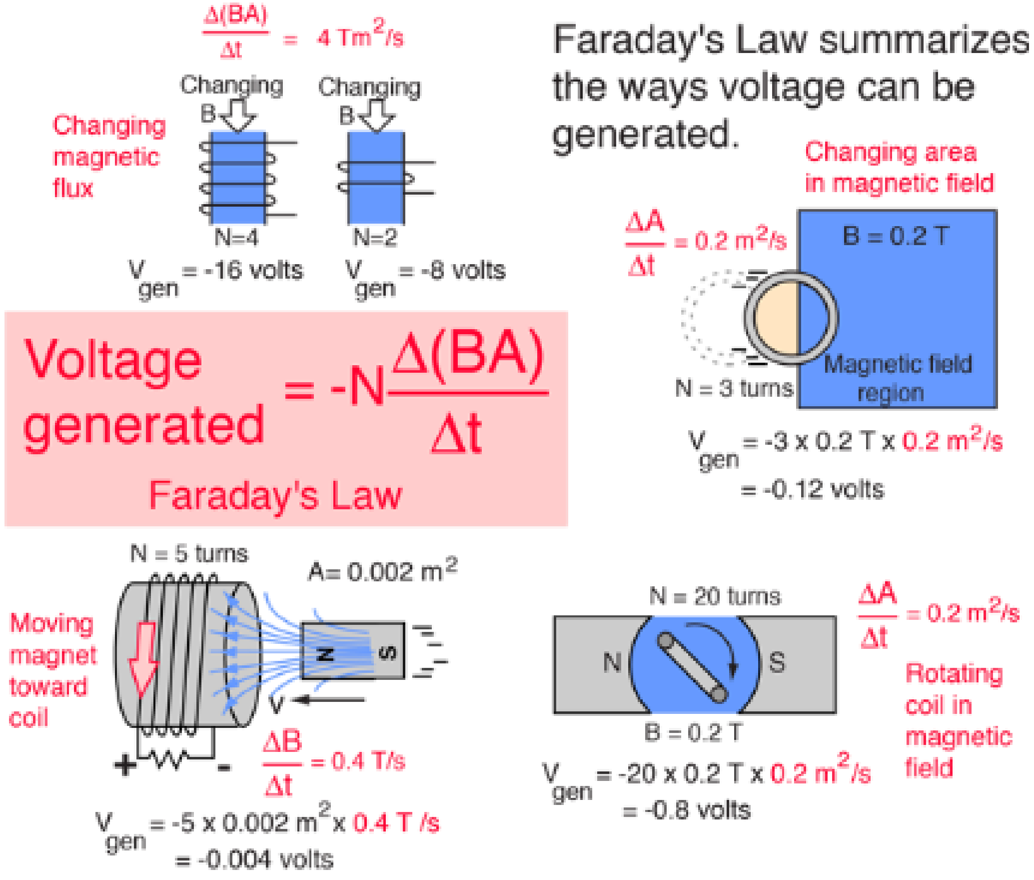
This law states that the Electric Flux out of a closed surface is proportional to the total charge enclosed by that surface

http://hyperphysics.phy-astr.gsu.edu/

Maxwell Equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$ Gauss Law of electricity

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ **Faraday's Law of induction**



Faraday's law states that when there is a change in magnetic flux (changing with respect to time) linking a coil or any conductor, there will be an EMF induced in the coil. Lenz's stated that the EMF induced will be in a direction such that it opposes the change in magnetic flux producing it.

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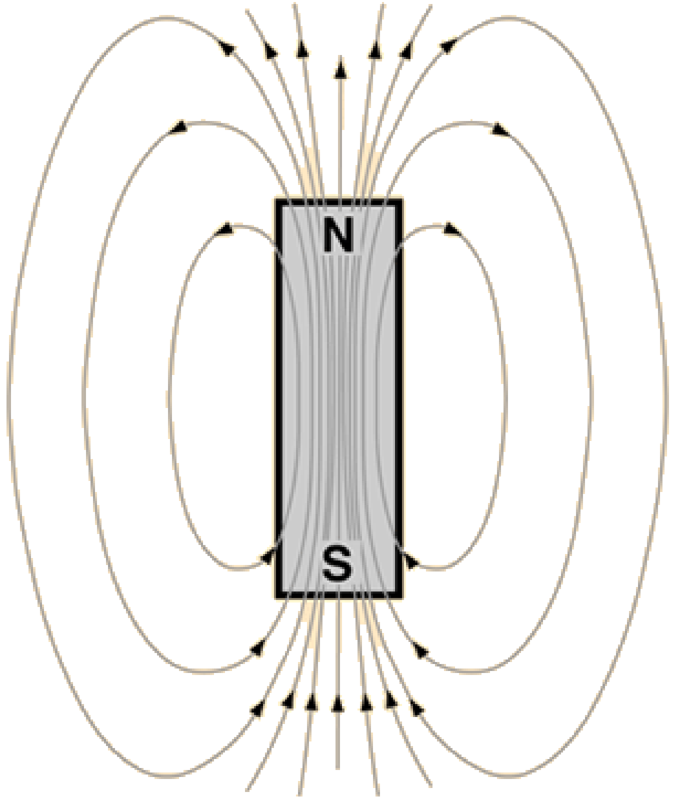
Maxwell Equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$ Gauss Law of electricity

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ Faraday's Law of induction

$$\nabla \cdot \boldsymbol{B} = 0$$ **Gauss Law of Magnetism**



Neither the north pole nor south pole individually acts a source or sink like the electric charges: magnetic monopoles do not exist

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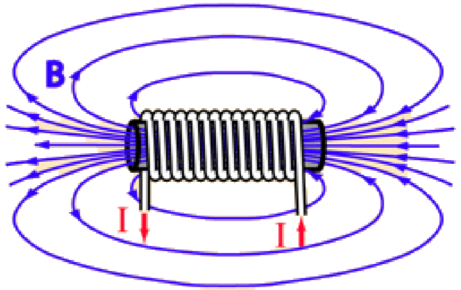
Maxwell Equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$ Gauss Law of electricity

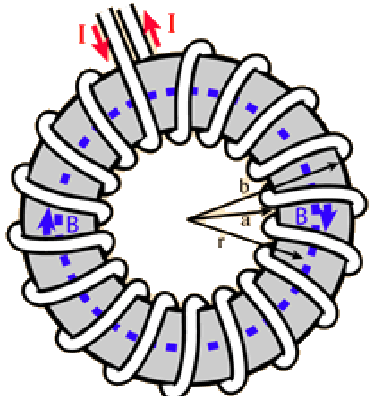
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ Faraday's Law of induction

$$\nabla \cdot \boldsymbol{B} = 0$$ Gauss Law of Magnetism

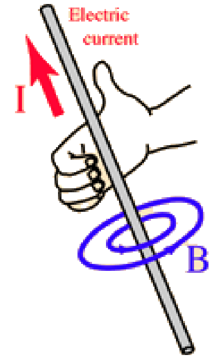
$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$ **Ampere's Law**



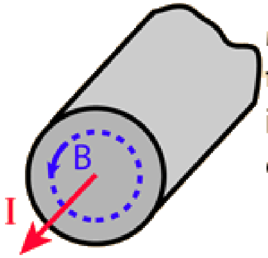
Magnetic field inside a long solenoic.



Magnetic field inside a toroidal coil.



Magnetic field from a long straight wire.



Magnetic field inside a conductor.

Ampere's law states that when an electric current flows through a wire, it produces a magnetic field around it

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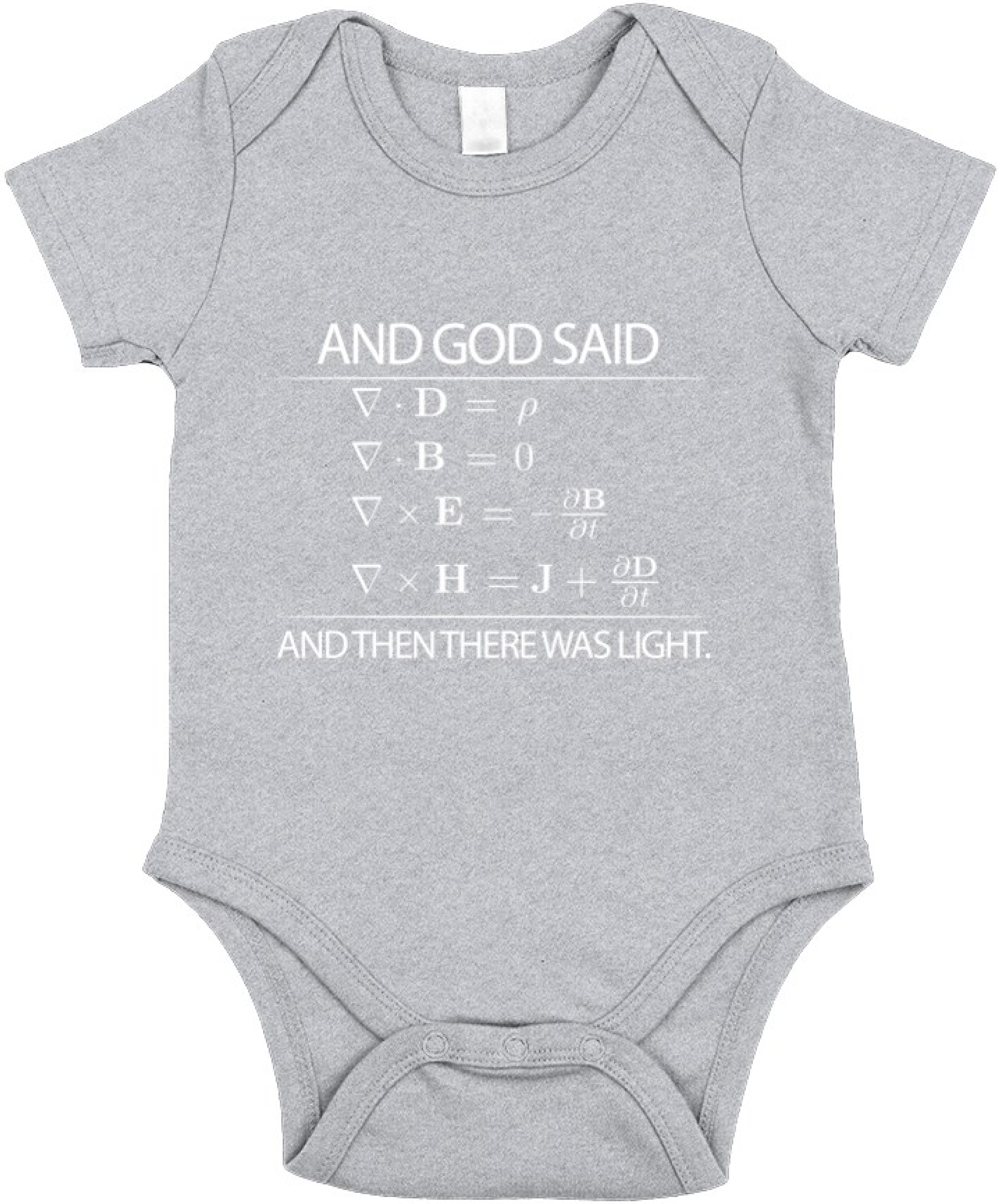
Maxwell Laws

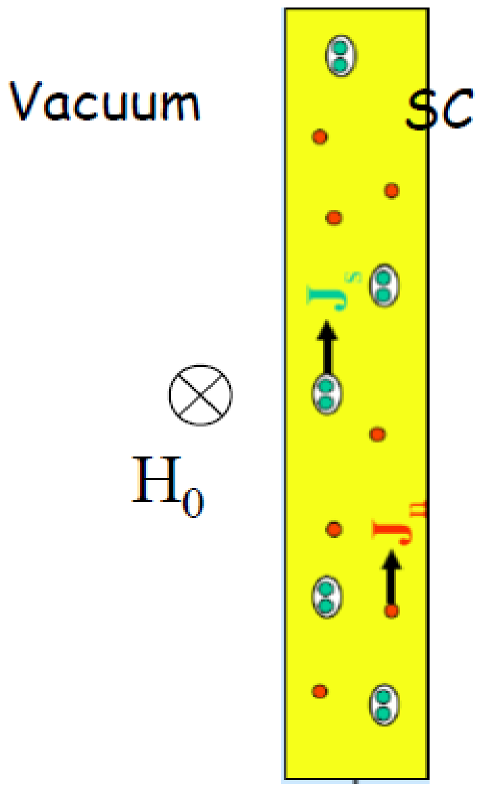


Two fluid model - Gorter and Casimir, 1934

Gorter and Casimir at Leiden developed this simple model to explain zero resistivity and perfect diamagnetism

Charge carriers are divided in two subsystems, superconducting carriers of density n_s and normal electrons of density n_n

$$\boldsymbol{n} = \boldsymbol{n_n} + \boldsymbol{n_s} \Rightarrow \boldsymbol{J} = \boldsymbol{J_n} + \boldsymbol{J_s}$$



G. Ciovati, Basic Principle of RF superconductivity, Tutorals SRF15

Two fluid model - Gorter and Casimir, 1934 (2)

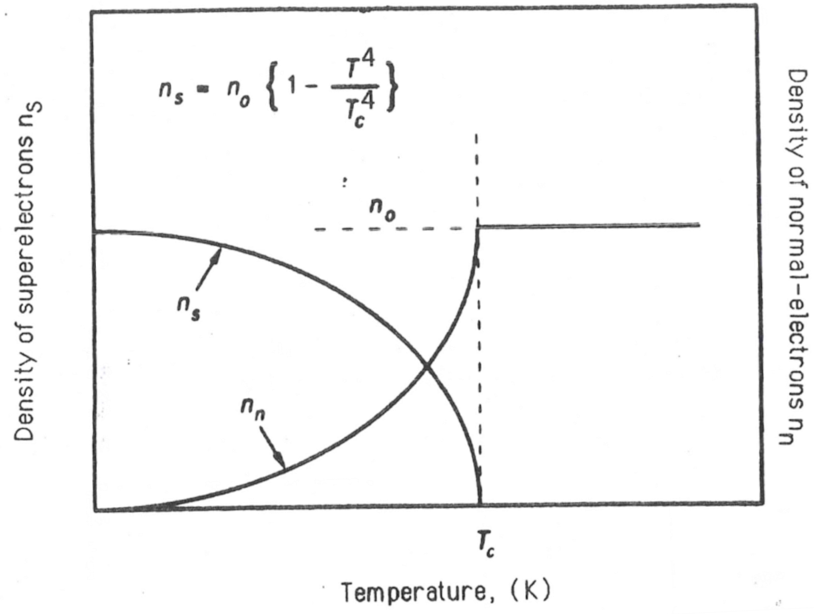
Gorter and Casimir at Leiden developed this simple model to explain zero resistivity and perfect diamagnetism

Charge carriers are divided in two subsystems, superconducting carriers of density n_s and normal electrons of density n_n

$$n = n_n + n_s \Rightarrow J = J_n + J_s$$

The number of SC electrons **depends by T**

$$n_s = n_0\left(1 + \frac{T^4}{T_c^4}\right)$$

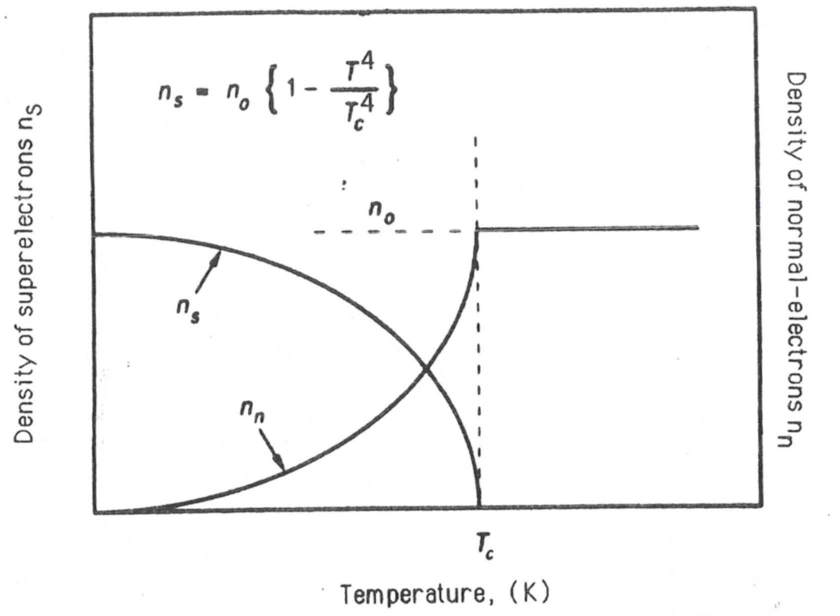


V. Palmieri, The classical superconductivity, 1992

Two fluid model - Gorter and Casimir, 1934 (2)

$$\boldsymbol{n = n_n + n_s} \quad \Rightarrow \quad \boldsymbol{J = J_n + J_s}$$

$$\boldsymbol{n_s = n_0\left(1 + \frac{T^4}{T_c^4}\right)}$$



V. Palmieri, The classical superconductivity, 1992

Two Fluid model is the fundament of each SC theory developed later

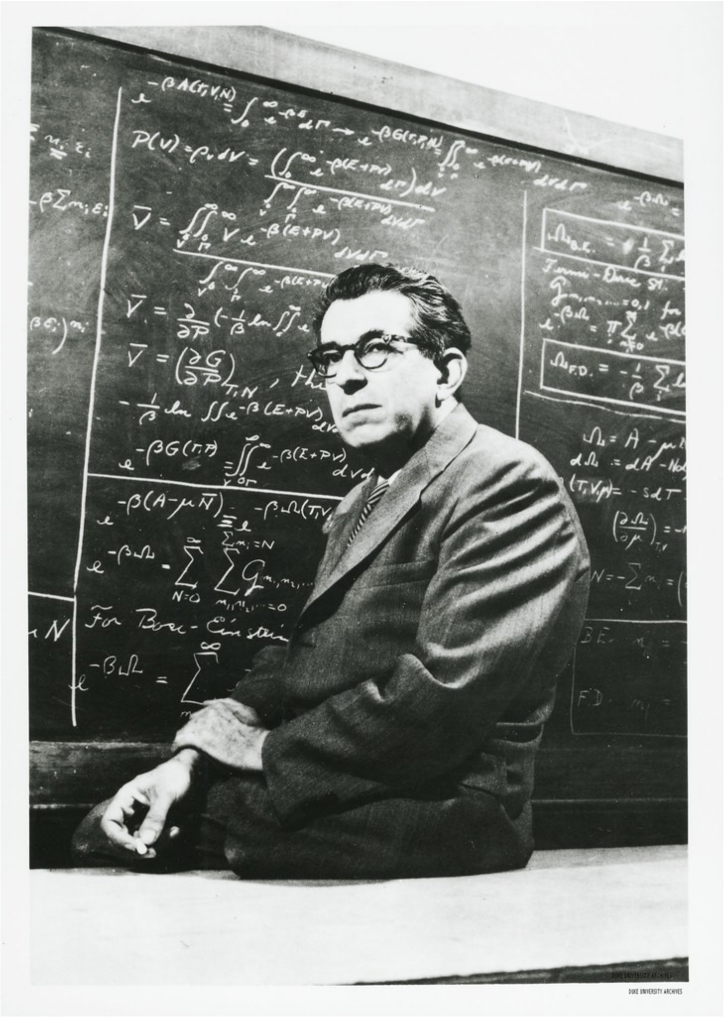
The London brothers

Fritz London is well known to material scientists…

1. **Valence Bond Theory** or Heitler–London Theory
   (chemical bonds in a molecule are overlapping of the atomic orbitals)

2. **London dispersion forces**
   (instantaneous dipole–induced dipole forces)

In his carrier has been worked with Sommerfield and Schoredinger among others…



Fritz London

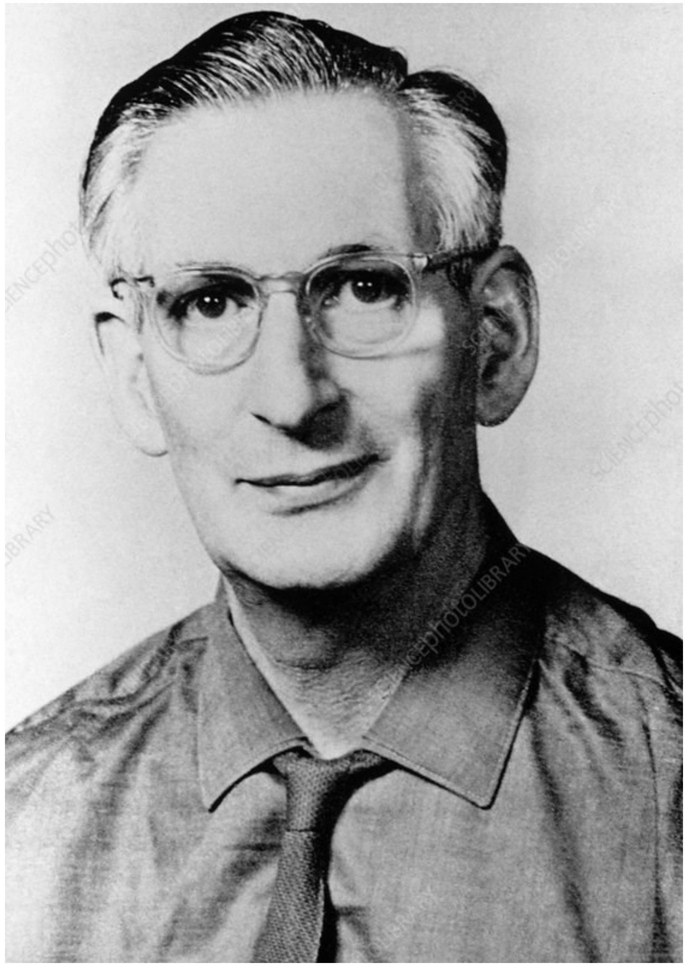
The London brothers

Heinz London (the younger brother) worked on Superconductivity with **Franz Simon** in **Breslau** (Germany)

Franz Simon was Jewish and in the thirties, since nazi-fascism was rising over Germany, moved to **Oxford** with his team, where **Frederick Lindemann** (head of the physics department) hired refugee scientists

Lindemann at that time had a second motivation in addition to the humanitarian one: reduce the gap with Cambridge's Cavendish Laboratory

Lindemann also wanted a theoretician to be added to the new SC group and Fritz London was the right person for him



Heinz London

The London brothers

Fritz London and his wife moved into a house in Hill Top Road, Oxford, and Heinz stayed with them, giving the brothers an opportunity to talk and work together about superconductivity

The London brother joint work was to provide the biggest breakthrough yet in understanding superconductivity

Heinz and Fritz London

Surface currents

Already years before the discovery of Meissner screening, speculations had been made by Onnes among others, that the **current in a superconductor might involve surface currents**

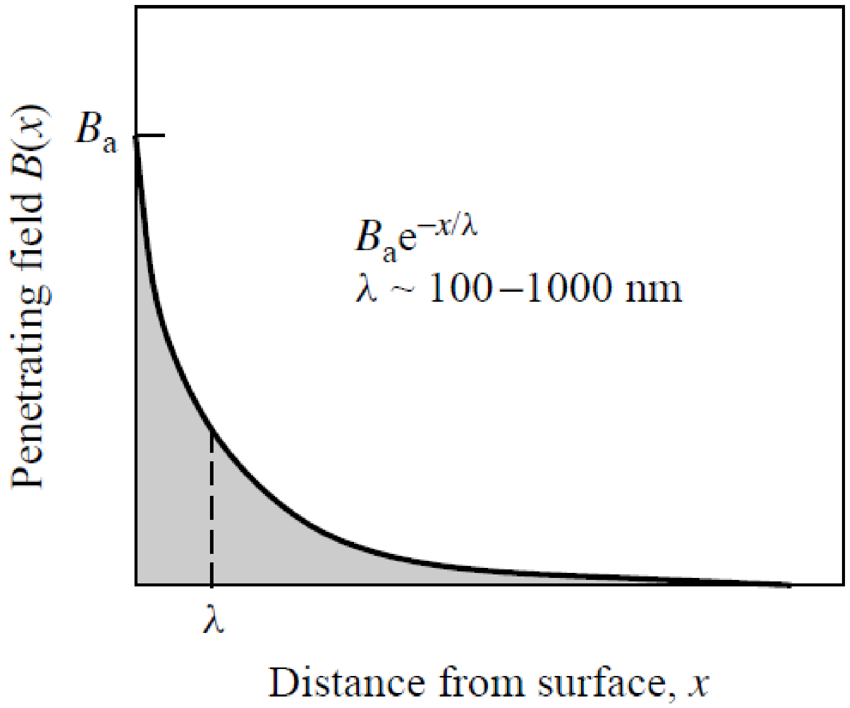
The discovery of Meissner and Ochsenfeld effect proved that this was correct

Logically the **magnetic field could be expelled** from the interior **only by setting up a surface current**

Penetration depth λ

Current could not exist only in the surface. It would have to **penetrate to a certain depth called (λ or λ_L)**

The penetration depth must be finite, and its length must be controlled by the free electron number density n_e. The **more electrons are present per volume, the more effective is the screening, and the shorter must λ be**



Derivation of 1st London equation

London brothers started with the logic that **E can accelerate superconducting electrons without friction/resistance**

$$m\frac{d\boldsymbol{v_s}}{dt} = e\boldsymbol{E}$$

since $\boldsymbol{J} = n_s e \boldsymbol{v_s}$

$$\boxed{\frac{d\boldsymbol{J_s}}{dt} = \frac{n_s e^2}{m}\boldsymbol{E}}$$

1. E = 0 → J_s goes on forever
2. E is required to maintain an AC current

Derivation of 2nd London equation

London expressed the electromagnetic field in terms of a **vector potential A**

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

From Maxwell: $\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$

$$\boldsymbol{E} = -\left|\frac{\partial \boldsymbol{A}}{\partial t}\right|$$

Derivation of 2nd London equations (2)

Combining $\boldsymbol{E} = -\left|\frac{\partial \boldsymbol{A}}{\partial t}\right|$ and $\frac{d\boldsymbol{J_s}}{dt} = \frac{n_s e^2}{m}\boldsymbol{E}$

$$\boldsymbol{J_s} = -\frac{n_s e^2}{m}\boldsymbol{A} \quad \xleftarrow{\text{INTEGRATING}} \quad \frac{d\boldsymbol{J_s}}{dt} = -\frac{n_s e^2}{m}\left|\frac{\partial \boldsymbol{A}}{\partial t}\right|$$

$$\boxed{\boldsymbol{J_s} = -\Lambda\,\boldsymbol{A}}$$

Analogous to Ohm's law: $\boldsymbol{J_n} = \sigma\boldsymbol{E}$

The factor Λ is a response function, analogous to the conductivity σ in a normal metal

Derivation of London penetration depth

$$\boldsymbol{J_s} = -\frac{n_s e^2}{m}\boldsymbol{A}$$

From the Amperes law $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J_s}$

we can obtain the expression for **B:**

$$\nabla \times (\nabla \times \boldsymbol{B}) = \mu_0 \nabla \times \boldsymbol{J_s}$$

$$\nabla \times (\nabla \times \boldsymbol{B}) = \cancel{\nabla(\nabla \cdot \boldsymbol{B})} - \nabla^2 \boldsymbol{B}$$

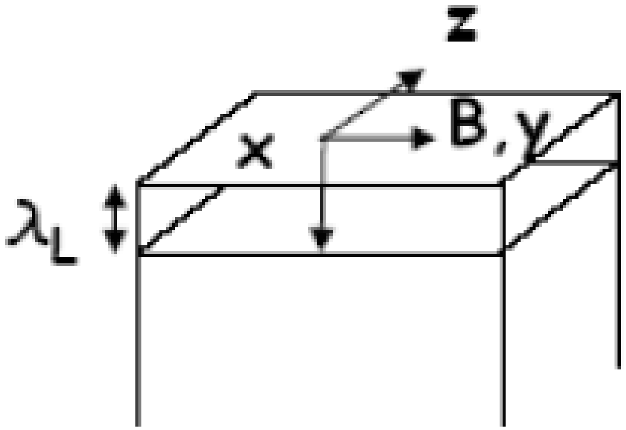
$$\mu_0 \nabla \times \boldsymbol{J_s} = -\mu_0 \left(\frac{n_s e^2}{m}\right) \underbrace{\nabla \times \boldsymbol{A}}_{\boldsymbol{B}}$$

Derivation of London penetration depth (2)

$$\nabla \times (\nabla \times \boldsymbol{B}) = \mu_0 \nabla \times \boldsymbol{J_s} \quad \Rightarrow \quad \nabla^2 \boldsymbol{B} = \mu_0 \left(\frac{n_s e^2}{m} \right) \boldsymbol{B}$$

The simplest situation is the 1-dimensional case with the field applied parallel to the y-axis along the surface of a superconductor of long length along z, and with the x-axis measuring the distance from the surface into the superconductor

$$\frac{\partial^2 B(x)}{\partial x^2} = \frac{\mu_0 n_s e^2}{m} B(x)$$



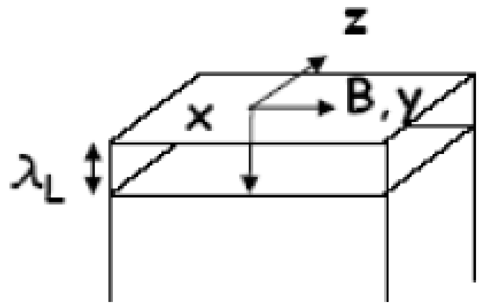
Derivation of London penetration depth (3)

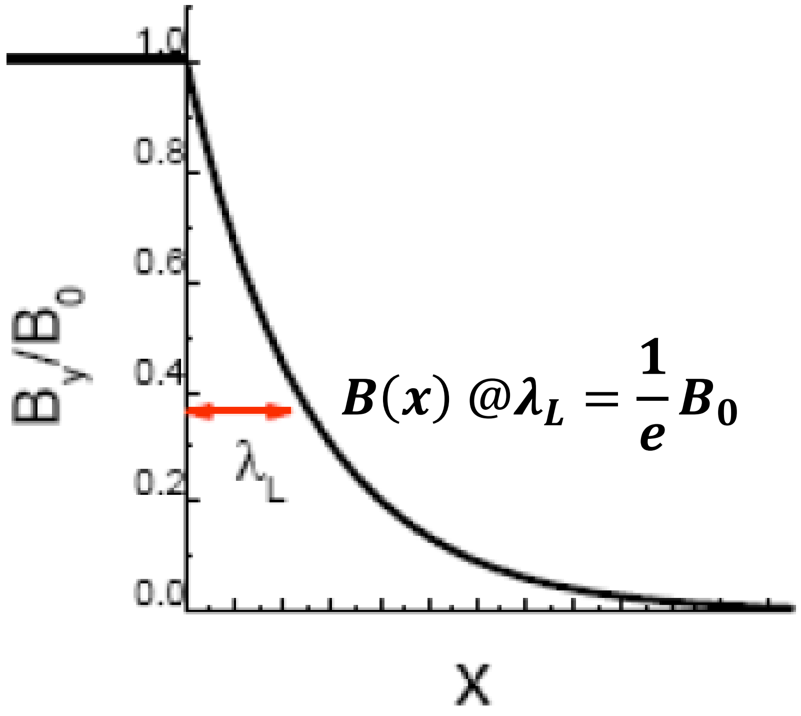
$$\frac{\partial^2 B(x)}{\partial x^2} = \frac{\mu_0 n_s e^2}{m} B(x)$$

The solution is: $\boldsymbol{B(x) = B_0 e^{-\left(\frac{x}{\lambda_L}\right)}}$

With **London penetration depth:**

$$\boldsymbol{\lambda_L^2 = \frac{1}{\mu_0}\frac{m}{n_s e^2}}$$



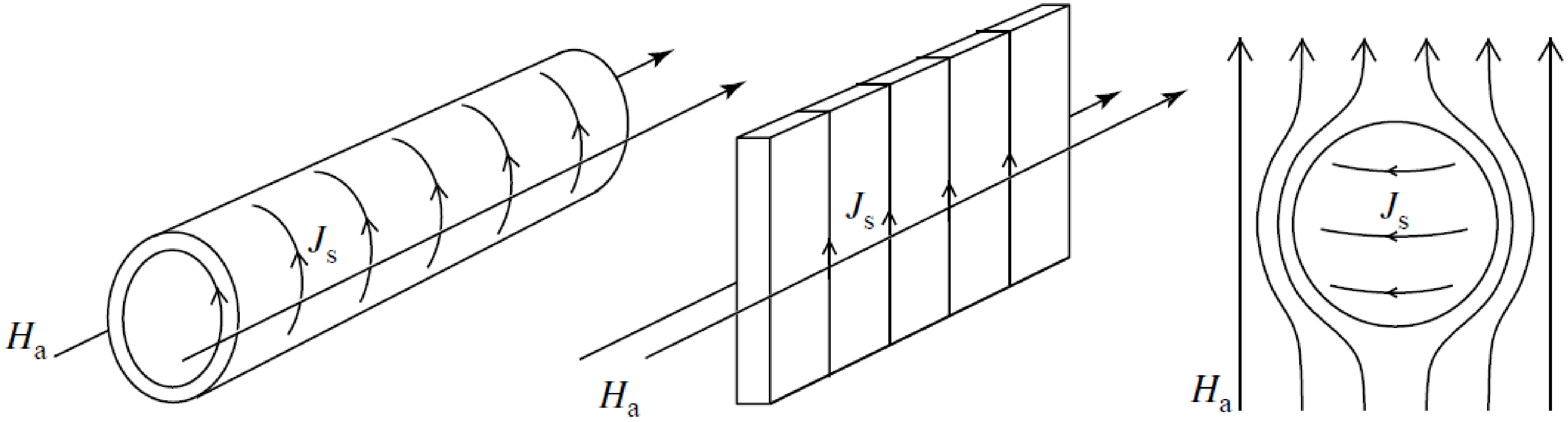


$\boldsymbol{B(x)\ @\lambda_L = \frac{1}{e} B_0}$

Derivation of London penetration depth (4)

$$B(x) = B_0 e^{-\left(\frac{x}{\lambda_L}\right)}$$

$$\lambda_L^2 = \frac{1}{\mu_0}\frac{m}{n_s e^2}$$



2^{nd} London equation in terms of B

$$J_s = -\frac{n_s e^2}{m} A$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad and \quad \boldsymbol{\lambda_L^2} = \frac{\boldsymbol{1}}{\boldsymbol{\mu_0}} \frac{\boldsymbol{m}}{\boldsymbol{n_s e^2}}$$

$$\nabla \times \boldsymbol{J_s} = -\frac{1}{\mu_0 \lambda_L^2} \boldsymbol{B}$$

1. B is the source of J_s
2. Spontaneus flux expulsion

London equations meaning

$$\frac{d\boldsymbol{J}_{\boldsymbol{s}}}{dt} = \frac{n_s e^2}{m}\boldsymbol{E}$$

Zero resistance

$$\nabla \times \boldsymbol{J}_{\boldsymbol{s}} = -\frac{1}{\mu_0 \lambda_L^2}\boldsymbol{B}$$

Meissner effect

Estimation of London penetration depth

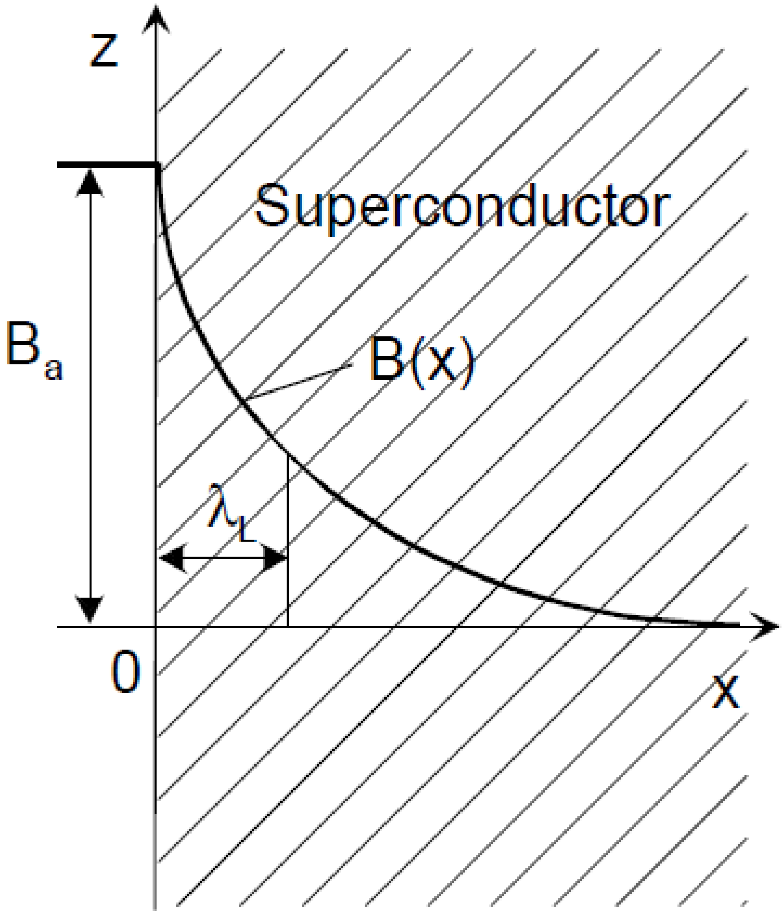
From $$\lambda_L^2 = \frac{1}{\mu_0}\frac{m}{n_s e^2}$$

we can obtain **a rough estimate of the London penetration depth** with the simplifying assumption that **one electron per atom with free-electron mass m_e contributes to the supercurrent**

For Sn, for example, such an estimate yields:

λ_L = 26 nm

This value deviates only little from the measured value, which at low T is in the range **25–36 nm**



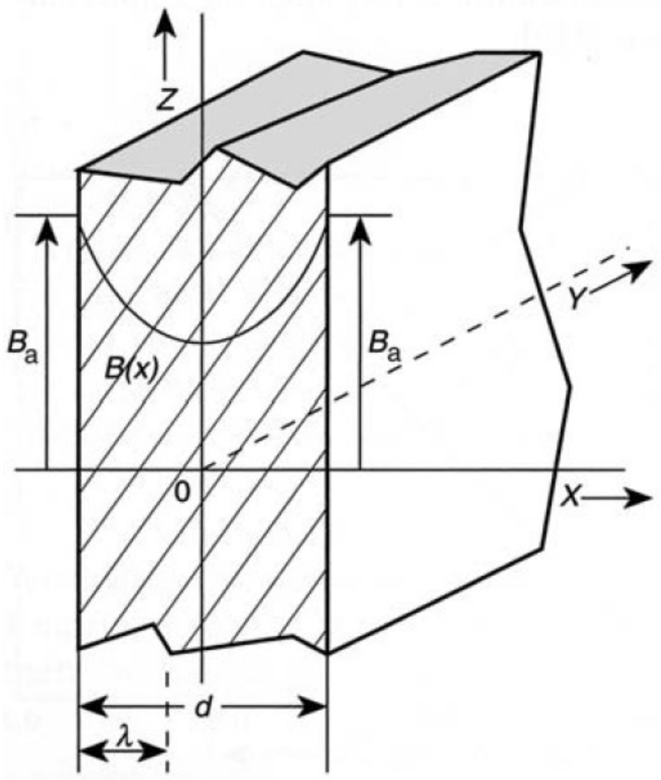
Measure of London penetration depth

How can one measure λ_L?

We can measure the **shielding effect** due to the **diamagnetic behavior** in **function of the thickness**

d >> λ_L → ***perfect magnetic shielding***

d ~ λ_L → ***poor magnetic shielding***



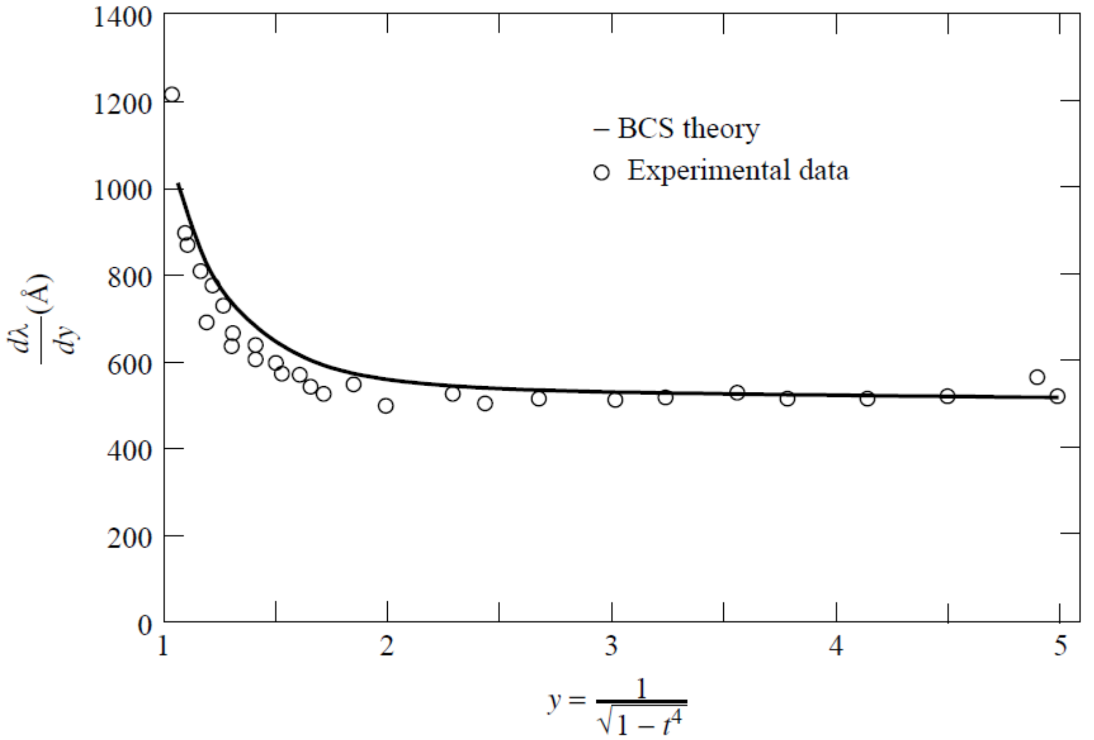
Spatial dependence of the magnetic field in a thin superconducting layer of thickness *d*. For the assumed ratio $d/\lambda_L = 3$, the magnetic field only decreases to about half of its outside value

London penetration depth T dependence

$$\lambda_L^2 = \frac{1}{\mu_0}\frac{m}{n_s e^2}$$

According with two-fluid model, n_s **depend on T**

$$\left(\frac{\lambda}{\lambda_0}\right)^2 = \frac{1}{1-\left(\frac{T}{T_c}\right)^4}$$



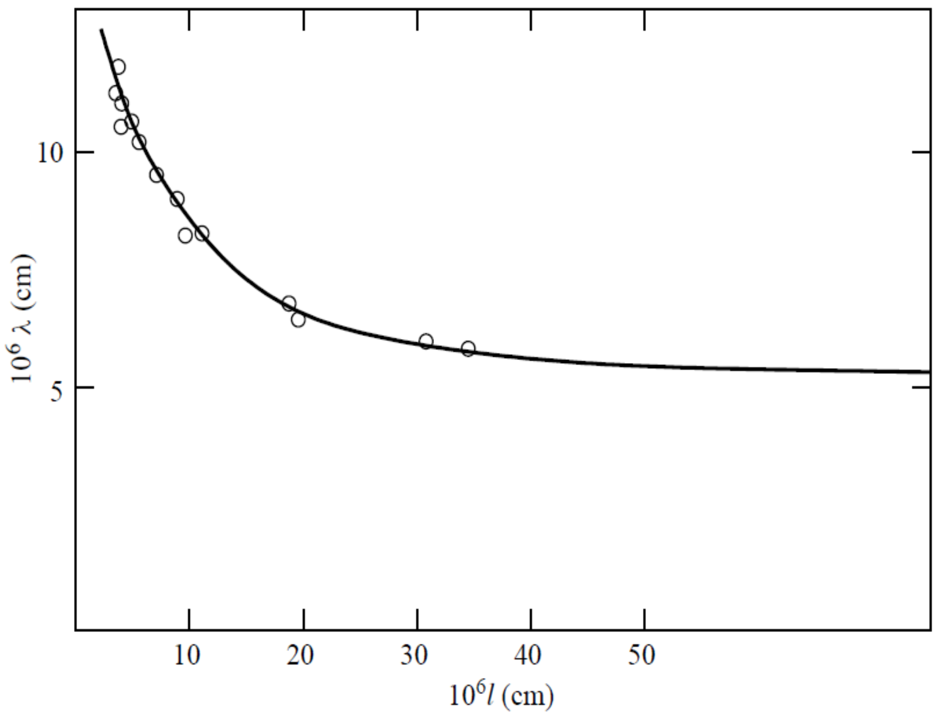
Early results on the temperature dependence of λ in a type I superconductor (Fosseim – Subdo)

Effect of impurities on λ_L

Pippard showed that **λ depended very sensitively on impurity scattering**

Addition of only 3% indium to tin λ changed by a factor of 2, while at the same time the **changes in T_c and H_c were insignificant**

In the **London theory λ depended only on m/n**, offering no hint at an explanation



Pippard non-local behaviour as evidenced by mean free path dependent penetration depth (Fossehim – Subdo)

Pippard - Non local version of London equation

In 1953, **Pippard** realized that the **key point** was a **breakdown** of the underlying assumption **of local response in the London theory**

In complete analogy with Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, which we pointed out is **formally equivalent to the London equation**, Pippard adopted the corresponding non-local version of Ohm's law from Reuther and Sondheimer for the so-called **anomalous skin-effect**

Brian Pippard

Electrodynamics of normal conductors

$$\boldsymbol{E} = \boldsymbol{E_0 e^{i\omega t}}$$

We can derive the **skin depth** starting from the fundamental equation of electrodynamics:

Maxwell's equations **+** *Linear and isotropic Material's equation* **+** *Drude's model*

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\boldsymbol{D} = \varepsilon_0 \varepsilon\, \boldsymbol{E}$$

$$\boldsymbol{B} = \mu_0 \mu\, \boldsymbol{H}$$

$$\boldsymbol{J} = \sigma\, \boldsymbol{E}$$

Skin depth

For a good conductor at RF frequencies: **ωε << σ** $\Rightarrow$ $\frac{\partial \boldsymbol{D}}{\partial t} \sim 0$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \cancel{\frac{\partial \boldsymbol{D}}{\partial t}} \Rightarrow \nabla \times \boldsymbol{H} = \boldsymbol{J} \xrightarrow[\boldsymbol{J} = \sigma \boldsymbol{E}]{\nabla \times} \nabla \times \nabla \times \boldsymbol{H} = \sigma \nabla \times \boldsymbol{E}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\boldsymbol{B} = \mu_0 \mu \, \boldsymbol{H}$$

$$\boldsymbol{\nabla^2 H = i\sigma\mu_0\mu\omega H}$$

$$\boldsymbol{H} = \boldsymbol{H_0 e^{i\omega t}}$$

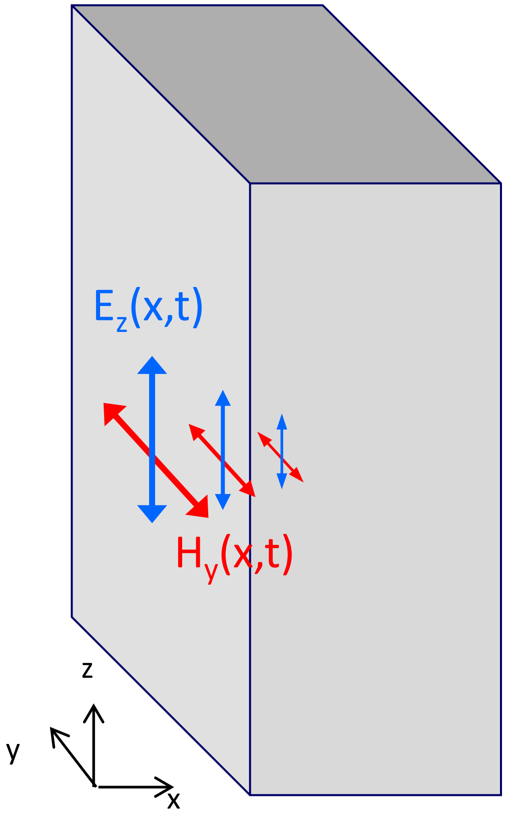
Skin depth (2)

$$\nabla^2 H = i\sigma\mu_0\mu\omega H$$

Solution (semi-infinite slab):

$$H_y = H_0 e^{-x/\delta} e^{-ix/\delta}$$

$$E_z = -\frac{(1+i)}{\sigma\delta} H_y$$



AC fields penetrate a thickness δ (the skin depth) $\delta = \sqrt{\frac{2}{\mu_0\mu\sigma\omega}}$

Surface impedence

$$Z = \frac{E_{\parallel}}{H_{\parallel}} = R_s + iX_s$$

Surface reactance

Surface resistance

For the semi-infinite plane conductor:

$$Z_n = \frac{|E_z|}{H_y} \xrightarrow{E_z = -\frac{(1+i)}{\sigma\delta} H_y} Z_n = \frac{1+i}{\sigma\delta}$$

$$R_s = X_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\mu_0 \mu \omega}{2\sigma}}$$

Anomalous skin effect

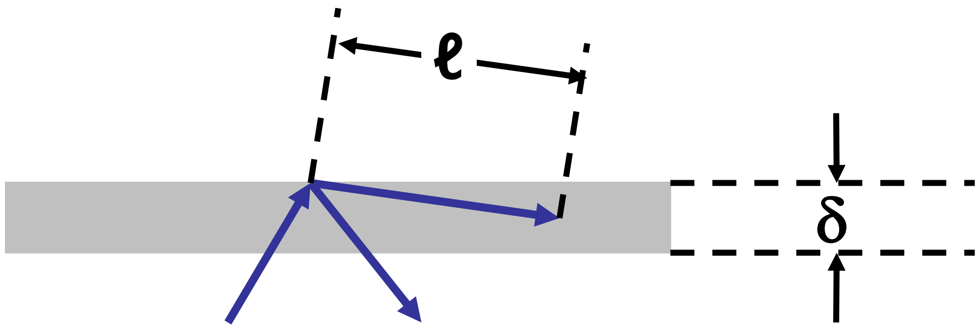
What happen at low T (and high frequency)?

$$R_s = \frac{1}{\sigma\delta}$$

σ(*T*) increases **δ** decreases

The skin depth (the distance over which fields vary) **can become less than the mean free path of the electrons** (the distance they travel before being scattered)

$$J(x) \neq \sigma\, E(x)$$



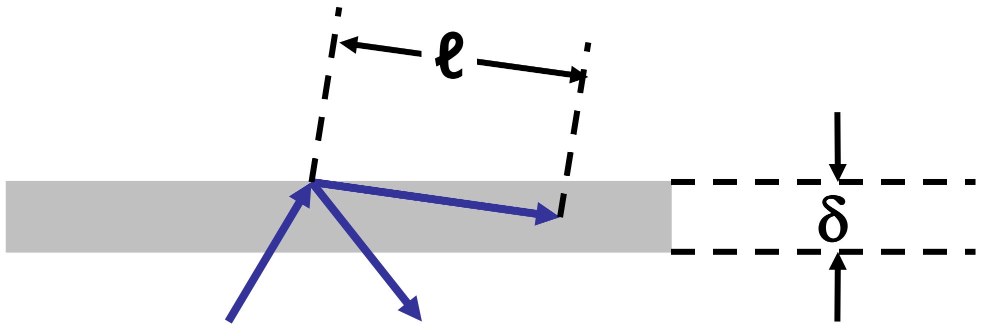
$$\delta = \sqrt{\frac{2}{\mu_0 \mu \sigma \omega}}$$

Anomalous skin effect (2)

Non local relationship introduced by Reuther and Sondheimer:

$$\boldsymbol{J} = \frac{3\sigma}{4\pi\ell}\int \frac{\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{E})e^{-r/\ell}}{r^4}d^3\boldsymbol{r}$$

Non-locality enters the problem when the response to a field can only be determined correctly by integrating over a volume of the size of ℓ^3 (3D case), where ℓ is comparable to or longer than the distance δ, the depth over which the $\boldsymbol{E}$-field varies



Surface resistance - some numbers

For Cu @ 300 K and 1.5 GHz:

σ (300 K) = 5.8 x 10^7 1/Ωm

μ_0=1.26x10^{-6} Vs/Am

μ=1

$$\delta = \sqrt{\frac{2}{\mu_0 \mu \sigma \omega}} = 1.7\ \mu\text{m}$$

$$R_s = \frac{1}{\sigma\delta} = 10\ \text{m}\Omega$$

Surface resistance - some numbers (2)

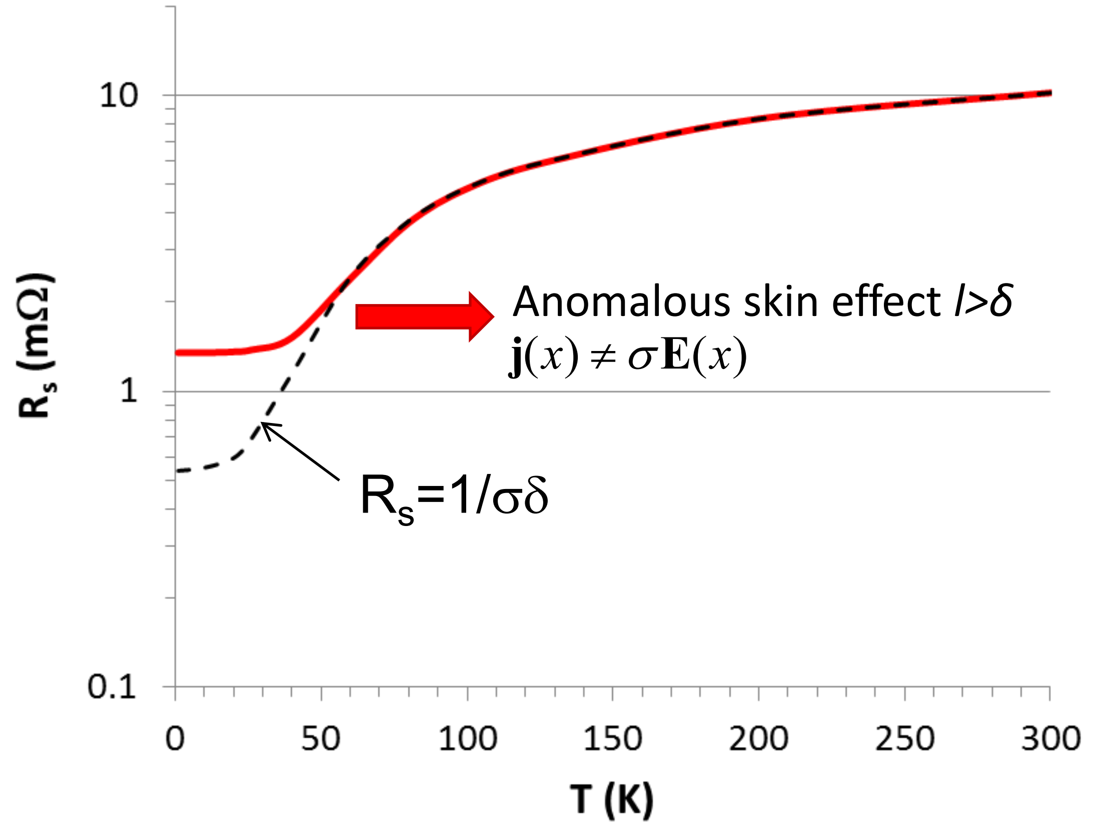
Surface resistance of Cu at 1.5 GHz as a function of temperature

R_s(300 K) $\cong$ 10 $\mathbf{m\Omega}$

R_s(4.2 K) $\cong$ 1.3 $\mathbf{m\Omega}$

RRR = σ(4.2K)/σ(300K) = 300

…in spite of the **resistivity decreasing by a factor 300** from 300 K to 4.2 K, **R_s only decreases by a factor of ~8**!



Pippard - Non local version of London equation

From experience in studies of the anomalous skin-effect in pure, normal metals **Pippard** saw the **analogy for the superconductor**, introducing the so called **coherence length ξ**

The **response** of the SC **to the applied magnetic field** is **nonlocal**, in the sense that the value of $\mathbf{J_s}$ measured at a point $\boldsymbol{r}$ depends on the value of $\boldsymbol{A}$ throughout a volume of radius ξ surrounding the point $\boldsymbol{r}$

Pippard - Non local version of London equation

In the simple case London equation $\boldsymbol{J_s} = -\wedge \boldsymbol{A}$ became:

$$\boldsymbol{J_s} = -\frac{\xi\,(\ell)}{\xi_0} \wedge \boldsymbol{A}$$

where $\boldsymbol{\xi_0}$ is the value of $\boldsymbol{\xi(\ell)}$ in the limit of large $\boldsymbol{\ell}$

and $\boldsymbol{\Lambda = (\mu_0 \lambda_L{}^2)^{-1}}$

The equation predicts a reduced response when ξ(l) is reduced by impurities

Pippard - Non local version of London equation (2)

Experimentally is found that:

$\ell \gg \xi$ *Clean limit*

$\ell \ll \xi$ *Dirty limit*

ℓ ↓ λ ↑ $\xi(\ell)$ ↓

In complete analogy with the non-local response to an electric field in a normal metal:

$$\boldsymbol{J} = \frac{3\sigma}{4\pi\ell}\int \frac{\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{E})e^{-r/\ell}}{r^4}d^3\boldsymbol{r} \Rightarrow \boldsymbol{J} = \frac{3\,\Lambda}{4\pi\xi_0}\int \frac{\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{A})e^{-r/\xi}}{r^4}d^3\boldsymbol{r}$$

where: $\xi_0 = a\dfrac{\hbar v_F}{k_B T_c}$

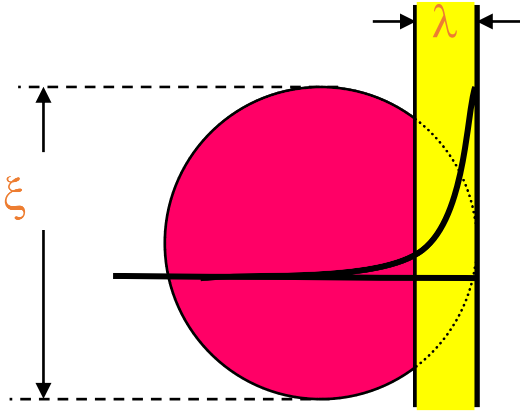
$a = 0{,}15$
$v_F = \text{Fermi velocity}$
$k_B = \text{Boltzmann constant}$
$T_c = \text{Critical Temperature}$

and: $\dfrac{1}{\xi(\ell)} = \dfrac{1}{\xi_0} + \dfrac{1}{\ell}$

Type I and Type II superconductors

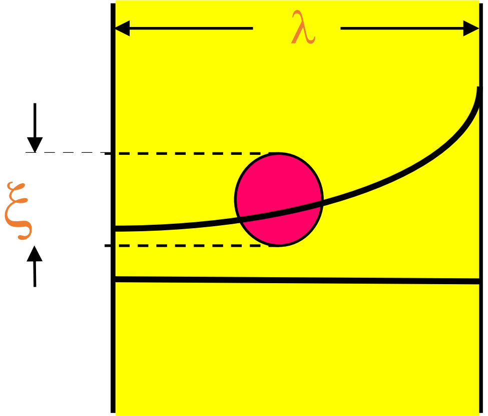
In the two opposite limits of $\xi \gg \lambda$ and $\xi \ll \lambda$

for $\xi \gg \lambda$ ➡ $$\lambda = \left(\frac{\sqrt{3}}{2\pi}\xi_0\lambda_L\right)^{\frac{1}{3}}$$



PIPPARD LIMIT, TYPE I SC ***(DIRTY LIMIT)***

for $\xi \ll \lambda$ ➡ $$\lambda = \frac{\xi_0}{\xi}\lambda_L$$

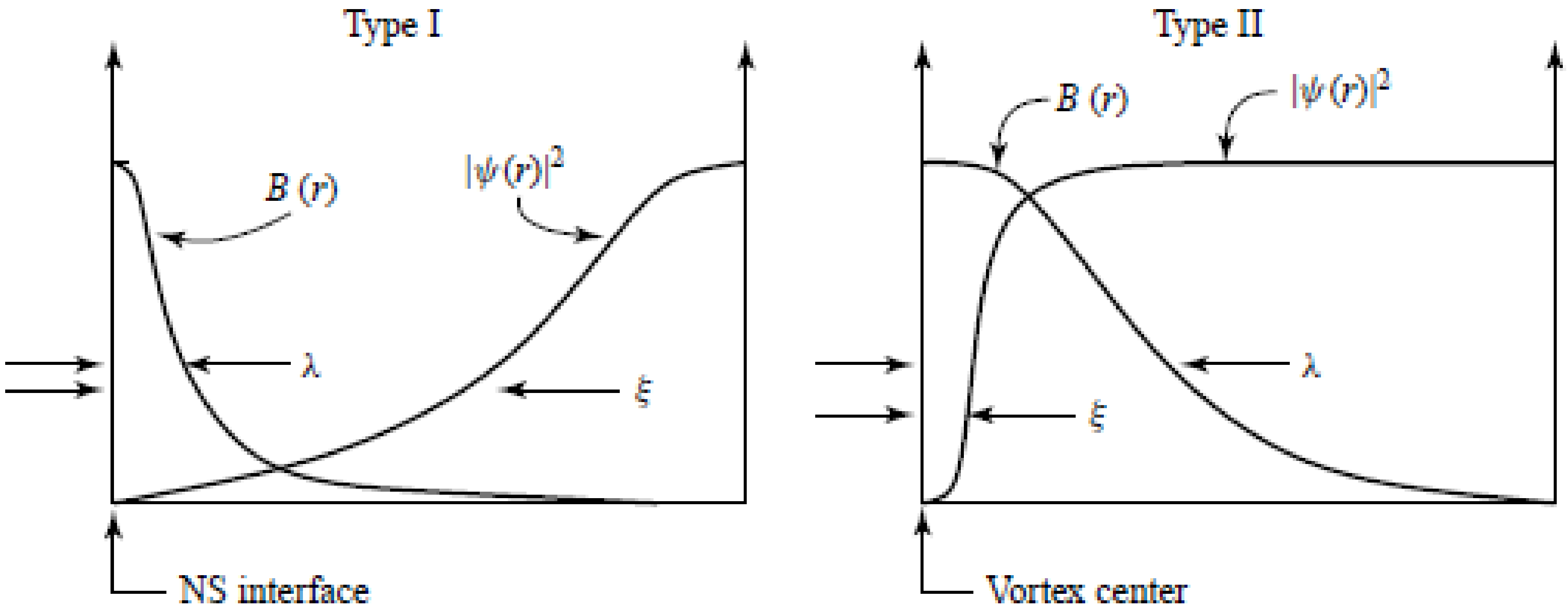


LONDON LIMIT, TYPE II SC ***(CLEAN LIMIT)***

Coherence length in BCS theory

BCS theory later confirmed the essential correctness of Pippard’s bold adaptation of non-local electrodynamics from normal metals to superconductors

In **BCS coherence length** is related to **characteristic Cooper pair size**



London's theory final remarks

London' theory explain infinite conductivity and Meissner effect

Expression of λ are only approximate

Calculates values of **λ differ from experimentally determined values**

This may to due to **uncertainty of the values of ns, e and m** taken for free electron

A **SC cannot be treated as a free electron metal**:
superelectrons in a SC interact coherently

Bibliography of this part

- K. Fossheim, A. Sudbø, "*Superconductivity - Physics and applications*", Wiley

  6.1 The London equation and the penetration depth λ_L

  7.3 Two tipes of superconductors

- W. Buckel, R. Kleiner, "*Superconductivity - Fundamentals and Applications*", Wiley

  1.4 Superconductivity: A Macroscopic Quantum Phenomenon