6° Tutorato

Ex. 3.

$$A_k = \begin{pmatrix} 1 & 2 & k \\ k-1 & 1 & 2 \\ 1 & -1 & 2-k \end{pmatrix}$$

(a) ? k NON iniettivo.

INIETTIVO

$\Updownarrow$

$\ker A_k = \{0\} \Rightarrow rk\, A_k = 3$.

$$\det A_k = \begin{vmatrix} 1 & 2 & k \\ k-1 & 1 & 2 \\ 1 & -1 & 2-k \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 2 \\ -1 & 2-k \end{vmatrix} - 2 \begin{vmatrix} k-1 & 2 \\ 1 & 2-k \end{vmatrix} + k \begin{vmatrix} k-1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 2-k+2-2\left[(k-1)(2-k)-2\right] +$$

$$+ k \left[\overbrace{-(k-1)-1}^{-k\, \cancel{+1}\, \cancel{-1}} \right] =$$

$$= 2 - k + 2 + 2\underbrace{(k-1)(k-2)}^{k^2-3k+2} + 4 - k^2 =$$

$$= k^2 - 7k + 12 = (k-3)(k-4).$$

$$\det A_k = 0 \iff \boxed{k=3} \vee \boxed{k=4}.$$

- $\boxed{k=3}$.

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\operatorname{Im} A_3 = \left\langle \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}} \right\rangle$$

$$\begin{cases} x + 2y + 3z = 0 \\ 2x + y + 2z = 0 \\ x - y - z = 0 \end{cases} \longrightarrow \operatorname{Ker} A_3 = \left\langle \underbrace{\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}} \right\rangle.$$

- $\boxed{k=4}$

$$A_4 = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\operatorname{Im} A_4 = \left\langle \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$$\ker A_4 \to \begin{cases} x+2y+4z=0 \\ 3x+y+2z=0 \\ x-y-2z=0 \end{cases}$$

$$\Rightarrow \ker A_4 = \left\langle \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle .$$

(c) A_k INIETTIVO $\Leftrightarrow k \neq 3,4$.

$$A_k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \qquad \downarrow$$

$\hookrightarrow$ INVERTIBILE $\Leftarrow \det A_k \neq 0$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A_k^{-1} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$A_k = \begin{pmatrix} 1 & 2 & k \\ k-1 & 1 & 2 \\ 1 & -1 & 2-k \end{pmatrix} \quad \det A_k = (k-3)(k-4) .$$

$$A_k^{-1} = \frac{1}{\det A_k} S_k^T$$

$$S_k = \begin{pmatrix} -(k-4) & (k+1)(k-4) & -k \\ k-4 & -2(k-1) & 3 \\ -(k-4) & (k+1)(k-2) & -2k+3 \end{pmatrix}$$

$$\delta_{11} = \begin{vmatrix} 1 & 2 \\ -1 & 2-k \end{vmatrix} = 2-k+2 = 4-k$$

$$-(k-1)(k-2)$$

$$\delta_{12} = -\begin{vmatrix} k-1 & 2 \\ 1 & 2-k \end{vmatrix} = -\left[(k-1)(2-k) - 2\right]$$

$$\overset{!}{=} +(k^2-3k+2)+2$$

$$\overset{!}{=} -\left[k^2+3k-4\right]$$

$$\overset{!}{=} k^2-3k+4 = (k+1)(k-4)$$

$$\delta_{13} = \begin{vmatrix} k-1 & 1 \\ 1 & -1 \end{vmatrix} = -k+\cancel{1}-\cancel{1} = -k$$

$$\delta_{21} = -\begin{vmatrix} 2 & k \\ -1 & 2-k \end{vmatrix} = -\left[4-2k+k\right]$$

$$\overset{!}{=} k-4$$

$$\delta_{22} = \begin{vmatrix} 1 & k \\ 1 & 2-k \end{vmatrix} = 2-k-k = -2k+2$$

$$\overset{!}{=} -2(k-1)$$

$$\delta_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -(-1-2) = 3$$

$$\delta_{31} = \begin{vmatrix} 2 & k \\ 1 & 2 \end{vmatrix} = 4 - k = -(k-4)$$

$$\delta_{32} = -\begin{vmatrix} 1 & k \\ k-1 & 2 \end{vmatrix} = -(2 - k(k-1))$$

$$= -2 + k^2 - k$$

$$= k^2 - k - 2 = (k+1)(k-2)$$

$$\delta_{33} = \begin{vmatrix} 1 & 2 \\ k-1 & 1 \end{vmatrix} = 1 - 2k + 2$$

$$= -2k + 3$$

$$A_k^{-1} = \begin{pmatrix} -\frac{1}{k-3} & \frac{1}{k-3} & \frac{-1}{k-3} \\ \frac{k^2-3k+4}{(k-3)(k-4)} & \frac{-2(k-1)}{(k-3)(k-4)} & \frac{(k+1)(k-2)}{(k-3)(k-4)} \\ -\frac{k}{\det A_k} & \frac{3}{\det A_k} & \frac{-2k+3}{\det A_k} \end{pmatrix}.$$

Ex. 4.

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

a) $$\det \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} \overset{?}{\neq} 0$$

$$\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2 - 2(2+1)$$

$$\overset{!}{=} 2 - 6 = -4 \neq 0 .$$

$\Longrightarrow \mathcal{A}$ è BASE di $\mathbb{R}^3$.

$$\det \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix} =$$

$$= 3 \begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} = 3(1+3) - 3$$

$$\overset{!}{=} 9 \neq 0 .$$

$\Longrightarrow \mathcal{B}$ è BASE di $\mathbb{R}^3$.

(b) $M_{A \to B} = ?$

$$\to M_{A \to \mathcal{E}} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$M_{\mathcal{E} \to A} = \left(M_{A \to \mathcal{E}}\right)^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}^{-1} =$$

$$= \frac{1}{-4} S^T$$

$$\left(S = \begin{pmatrix} 2 & -3 & -1 \\ -4 & 2 & 2 \\ -2 & 1 & -1 \end{pmatrix} \right.$$

$$= -\frac{1}{4} \begin{pmatrix} 2 & -4 & -2 \\ -3 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1 & 1/2 \\ 3/4 & -1/2 & -1/4 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}$$

$$\to M_{B \to \mathcal{E}} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix}$$

$$M_{\mathcal{E} \to B} = \left(M_{B \to \mathcal{E}}\right)^{-1} =$$

$$= \frac{1}{9} S^T \qquad S = \begin{pmatrix} 4 & 1 & 3 \\ -3 & -3 & -9 \\ -1 & 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4/9 & -1/3 & -1/9 \\ 1/9 & -1/3 & 2/9 \\ 1/3 & -1 & -1/3 \end{pmatrix}$$

$$M_{A \to B} = M_{\mathcal{E} \to B} \; M_{A \to \mathcal{E}}$$

$$= \begin{pmatrix} 4/9 & -1/3 & -1/9 \\ 1/9 & -1/3 & 2/9 \\ 1/3 & -1 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 5/9 & 1/9 \\ 0 & -1/9 & 7/9 \\ -1 & -1/3 & 1/3 \end{pmatrix}$$

$$M_{B \to A} = M_{\mathcal{E} \to A} \; M_{B \to \mathcal{E}} =$$

$$= \begin{pmatrix} -1/2 & 1 & 1/2 \\ 3/4 & -1/2 & -1/4 \\ 1/4 & -1/2 & 1/4 \end{pmatrix} \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & 1/2 & -1 \\ 7/4 & -1/4 & 0 \\ 1/4 & 5/4 & 0 \end{pmatrix}.$$