# 4° TUTORATO

$\boxed{\text{Def}}$

Un'applicazione $L: V \to W$ tra due spazi vettoriali $V$ e $W$ si dice <u>lineare</u> se:

(1) $\to \forall \bar{v}_1, \bar{v}_2 \in V, \; L(\bar{v}_1 + \bar{v}_2) = L(\bar{v}_1) + L(\bar{v}_2)$

(2) $\to \forall \bar{v} \in V, \; \forall \alpha \in \mathbb{R}, \; L(\alpha v) = \alpha L(v)$

(1) $f: \mathbb{R}^2 \xrightarrow{} \mathbb{R}^3$ (ENT $\mathbb{R}^2$, USCITA $\mathbb{R}^3$) $: (x, y) \to (2x, y, x-y)$

(1) $\to f(\bar{v}) + f(\bar{v}') = (2x, y, x-y) + (2x', y', x'-y')$

$\bar{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \bar{v}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$= (2x + 2x', y + y', x - y + x' - y')$

$= f(\bar{v} + \bar{v}')$ ✓

(2) $\to f(h\bar{v}) = f(hx, hy) \Rightarrow (2hx, hy, h(x-y))$

$= h f(\bar{v})$

$f(\bar{v}) = h(2x, y, x-y)$

**Def**

Il <u>nucleo</u> di una funzione lineare $L: V \to W$ è

$\ker L = \{ \bar{v} \in V \mid L(\bar{v}) = \bar{0}_W \}$

$\ker f : \{ \bar{v} \in \mathbb{R}^2 \mid f(\bar{v}) = (0, 0, 0) \}$

$= \{ (x, y) \in \mathbb{R}^2 \mid f(x, y) = (2x, y, x-y) = (0, 0, 0) \}$

$\begin{cases} 2x = 0 \\ y = 0 \\ x - y = 0 \end{cases} \qquad \begin{matrix} x = 0 \\ y = 0 \end{matrix} \qquad \Rightarrow \ker f = \{ 0, 0 \}$

È iniettiva?

Teorema $L: V \to W$

$L$ iniettiva $\iff \ker L = \{\bar{0}_V\}$

$\Rightarrow f$ è iniettiva ✓

TUTORATO 13/04.

Ex. 1

$\phi : \mathbb{R}^4 \to \mathbb{R}^3$

$$\phi(x,y,z,w) = \begin{pmatrix} x+y+2z \\ y-z+w \\ 2x+y-w \end{pmatrix}$$

(a) $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}$

- $\phi(v_1+v_2) = \phi(v_1) + \phi(v_2)$
- $\phi(\lambda v) = \lambda \phi(v)$ $\quad v = (x, y, z, w)$

$$\phi(v_1+v_2) = \phi\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \\ w_1+w_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2+y_1+y_2+2(z_1+z_2) \\ y_1+y_2-(z_1+z_2)+w_1+w_2 \\ 2(x_1+x_2)+y_1+y_2-(w_1+w_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1+y_1+2z_1 \\ y_1-z_1+w_1 \\ 2x_1+y_1-w_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2+2z_2 \\ y_2-z_2+w_2 \\ 2x_2+y_2-w_2 \end{pmatrix} = \phi(v_1)+\phi(v_2)$$

$$\phi(\lambda v) = \phi\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda w \end{pmatrix} = \begin{pmatrix} \lambda x+\lambda y+2\lambda z \\ \lambda y-\lambda z+\lambda w \\ 2\lambda x+\lambda y-\lambda w \end{pmatrix} = \lambda\phi(v)$$

$\Longrightarrow \phi$ LINEARE

(b) $v \in \operatorname{Ker} \phi \Longleftrightarrow \phi(v) = 0$

$$\begin{cases} x + y + 2z = 0 \\ y - z + w = 0 \longrightarrow z = y + w = 2x + 2y \\ 2x + y - w = 0 \longrightarrow w = 2x + y \end{cases}$$

$$x + y + 4x + 4y = 0$$

$$5x = -5y \longrightarrow y = -x$$

$$\begin{cases} y = -x \\ z = 0 \\ w = x \end{cases} \qquad \operatorname{Ker} \phi = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\dim(\mathbb{R}^4) = 4 = \underset{\underset{1}{\parallel}}{\dim \operatorname{Ker} \phi} + \dim \operatorname{Im} \phi \implies \dim \operatorname{Im} \phi = 3$$

$$\phi \longleftrightarrow A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 \end{pmatrix} = M(\phi)^{\mathcal{E}}$$

$$\operatorname{Im} \phi = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

(c)

$$\mathcal{V} = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}, \overset{v_4}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}} \right\}$$ DOMINIO

$$\mathcal{W} = \left\{ \underset{w_1}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}, \underset{w_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \right\}$$ CODOMINIO

$$M(\phi)^{\mathcal{W}}_{\mathcal{V}} = ?$$

$$\phi(e_1) = e_1 + 2e_3$$
$$\phi(e_2) = e_1 + e_2 + e_3$$
$$\phi(e_3) = 2e_1 - e_2$$
$$\phi(e_4) = e_2 - e_3$$

$$v_1 = e_1 + e_2$$
$$v_2 = e_2 + e_3$$
$$v_3 = e_3 + e_4$$
$$v_4 = e_2$$

$$\phi(v_1) = \phi(e_1 + e_2) = \phi(e_1) + \phi(e_2) =$$
$$= e_1 + 2e_3 + e_1 + e_2 + e_3 = 2e_1 + e_2 + 3e_3$$
$$\phi(v_2) = \phi(e_2 + e_3) = \phi(e_2) + \phi(e_3) =$$
$$= e_1 + \cancel{e_2} + e_3 + 2e_1 - \cancel{e_2} = 3e_1 + e_3$$

$$\phi(v_3) = \phi(e_3 + e_4) = \phi(e_3) + \phi(e_4) =$$

$$= 2e_1 - \cancel{e_2} + \cancel{e_2} - e_3 = 2e_1 - e_3$$

$$\phi(v_4) = \phi(e_2) = e_1 + e_2 + e_3$$

$$\begin{cases} w_1 = e_1 + 2e_2 + e_3 \\ w_2 = -e_1 + e_2 + e_3 \\ w_3 = e_3 \end{cases}$$

$$\begin{cases} e_3 = w_3 \\ e_1 = e_2 + w_3 - w_2 \\ w_1 = e_2 + w_3 - w_2 + 2e_2 + w_3 \end{cases}$$

$$\hookrightarrow 3e_2 = w_1 + w_2 - 2w_3$$

$$\rightarrow e_2 = \frac{1}{3}w_1 + \frac{1}{3}w_2 - \frac{2}{3}w_3$$

$$\rightarrow e_1 = \frac{1}{3}w_1 - \frac{2}{3}w_2 + \frac{1}{3}w_3$$

$$\phi(v_1) = 2e_1 + e_2 + 3e_3$$

$$= \frac{2}{3}w_1 - \frac{4}{3}w_2 + \cancel{\frac{2}{3}w_3} + \frac{1}{3}w_1 + \frac{1}{3}w_2 - \cancel{\frac{2}{3}w_3} + 3w_3$$

$$= w_1 - w_2 + 3w_3$$

$$\phi(v_2) = 3e_1 + e_3$$

$$= w_1 - 2w_2 + w_3 + w_3 = w_1 - 2w_2 + 2w_3$$

$$\phi(v_3) = 2e_1 - e_3$$
$$= \frac{2}{3}w_1 - \frac{4}{3}w_2 + \frac{2}{3}w_3 - w_3$$
$$= \frac{2}{3}w_1 - \frac{4}{3}w_2 - \frac{1}{3}w_3$$

$$\phi(v_4) = e_1 + e_2 + e_3$$
$$= \frac{1}{3}w_1 - \frac{2}{3}w_2 + \frac{1}{3}w_3 + \frac{1}{3}w_1 + \frac{1}{3}w_2 - \frac{2}{3}w_3 +$$
$$+ w_3$$
$$= \frac{2}{3}w_1 - \frac{1}{3}w_2 + \frac{2}{3}w_3$$

$$M(\phi)_v^w = \begin{pmatrix} 1 & 1 & 2/3 & 2/3 \\ -1 & -2 & -4/3 & -1/3 \\ 3 & 2 & -1/3 & 2/3 \end{pmatrix}$$

(d) ? $\exists\, W \subset \mathbb{R}^4$ t.c. $\dim(\phi(W)) = \dim(W)$

$W = \langle e_1, e_2, e_3 \rangle$ $\dim W = 3$

$\phi(e_1) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\phi(e_2) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\phi(e_3) = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$$\phi(W) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

$\dim(\phi(W)) = 3.$