III Tutorato

Def V spazio vettoriale

Un insieme finito di vettori $\{v_1, \dots, v_n\}$ di V si dice base di V se

(i) i vettori sono linearmente indipendenti

(ii) i vettori generano V

Definiamo $\dim V = n =$ vettori in una qualsiasi base

Formula di Grassman

Dati U, W sottospazi di V

$$\dim(V+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Ex 1

Dati i vettori

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

stabilire se sono una base di $\mathbb{R}^3$

come numero ci siamo => sono 3

Verifichiamo se sono lin dipendenti

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \overset{?}{\Longleftrightarrow} \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$\begin{cases} \lambda_1 + 3\lambda_2 - \lambda_3 = 0 \\ \lambda_1 + 0 + \lambda_3 = 0 \\ \lambda_1 - \lambda_2 - \lambda_3 = 0 \end{cases} \Longleftrightarrow \begin{cases} \lambda_1 + 6\lambda_1 + \lambda_1 = 0 \\ \lambda_3 = -\lambda_1 \\ \lambda_2 = \lambda_1 + \lambda_1 = 2\lambda_1 \end{cases} \Longleftrightarrow \begin{cases} 8\lambda_1 = 0 \\ \lambda_3 = -\lambda_2 \\ \lambda_2 = 2\lambda \end{cases}$$

$$8\lambda_1 = 0 \Longleftrightarrow \lambda_1 = 0 \Rightarrow \lambda_3 = 0 \Rightarrow \lambda_2 = 0$$

sono lin. dip => Base di $\mathbb{R}^3$

Ex 2

Stabilire se S è un sottospazio e trovarne la dimension

$$S = \left\{ \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} \middle| \; a, b \in \mathbb{R} \right\}$$

mettiamo insieme le 2 condizioni

Dati $\bar{u}, \bar{v} \in S$ e $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\bar{u} = \begin{pmatrix} a_1 \\ b_1 \\ a_1+b_1 \end{pmatrix} \qquad \bar{v} = \begin{pmatrix} a_2 \\ b_2 \\ a_2+b_2 \end{pmatrix}$$

$\lambda_1 \bar{u} + \lambda \bar{v} \in S$?

$$\Downarrow$$

$$\begin{pmatrix} \lambda_1 a_1 + \lambda_2 a_2 \\ \lambda_1 b_1 + \lambda_2 b_2 \\ \lambda_1 a_1 + \lambda_1 b_1 + \lambda_2 a_2 + \lambda_2 b_2 \end{pmatrix} \in S \quad \checkmark$$

$\Rightarrow$ È UN SOTTOSPAZIO

prendiamo una base qualsiasi

tipo $a = 1 \quad b = 0 \quad$ e $a = 0$ e $b = 1$

$$S = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \dim 2$$

~~Stabili~~

Ex 3

Stabilire se S è un sottospazio e trovarne la dim.

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in M_{3,1}(\mathbb{R}) \;\middle|\; 2x + z = 0 \right\}$$

$\Downarrow z = -2x$

$$\bar{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ -2x_1 \end{pmatrix} \qquad \bar{v}_2 = \begin{pmatrix} x_2 \\ y_2 \\ -2x_2 \end{pmatrix}$$

$$\lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2 = \begin{pmatrix} \lambda_1 x_1 + \lambda_2 x_2 \\ \lambda_1 y_1 + \lambda_2 y_2 \\ -2\lambda_1 x_1 - 2\lambda x_2 \end{pmatrix} \in S$$

$\Rightarrow$ è un SOTTOSPAZIO

$x = 0 \quad y = 1$

$x = 1 \quad y = 1$

$$S = \left\langle \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \Rightarrow \boxed{\dim 2}$$

Def.

Siano W_1 e W_2 sottospazi di uno spazio vettoriale V_K. La somma $W_1 + W_2$ si dice diretta se $\forall\, \bar{u} \in W_1 + W_2$ sono unici i vettori $\bar{w}_1 \in W_1$ e $\bar{w}_2 \in W_2$ tali che

$\bar{u} = \bar{w}_1 + \bar{w}_2$

$$\Downarrow$$

Teorema Siano W_1 e W_2 sottospazi di uno spazio vettoriale V_K. La somma di $W_1 + W_2$ è diretta

$\iff W_1 \cap W_2 = \{\bar{0}\}$

Ex 4

Dati i sottospazi di $\mathbb{R}^4$

$$U = \left\langle \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \right\rangle \qquad V = \left\langle \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\rangle$$

$U + V$ è diretta ?

$\bar{w} \in U \cap V \iff$

$$\bar{w} = \lambda_1 \begin{pmatrix}1\\0\\1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \mu_1 \begin{pmatrix}0\\1\\1\\1\end{pmatrix} + \mu_2 \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ 2\lambda_2 = \mu_1 \\ \lambda_1 + 3\lambda_2 = \mu_1 \\ 4\lambda_2 = \mu_1 + \mu_2 \end{cases} \iff \begin{cases} \lambda_2 = -\lambda_1 \\ 2(-\lambda_1) = \mu_1 \\ \lambda_1 + 3(-\lambda_1) = -2\lambda_1 \\ 4(-\lambda_1) = -2\lambda_1 + \mu_2 \end{cases}$$

$$\iff \begin{cases} \lambda_2 = -\lambda_1 \\ \mu_1 = -2\lambda_1 \\ -2\cancel{\lambda_1} = -\cancel{\lambda_1}\cdot 2 \quad \boxed{1=1} \\ -4\lambda_1 + 2\lambda_1 = \mu_2 \end{cases} \iff \begin{cases} \lambda_2 = -\lambda_1 \\ \mu_1 = -2\lambda_1 \\ \mu_2 = -2\lambda_1 \\ \lambda_1 = \lambda_1 \end{cases}$$

qualsiasi

$\boxed{\lambda = 1}$

$$U \cap V = \left\langle \begin{pmatrix}1-1\\-2\\1-3\\-4\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}0\\-2\\-2\\-4\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}0\\1\\1\\2\end{pmatrix} \right\rangle$$

$\neq \underline{0} \Rightarrow$ NON È DIRETTA!

65| Determinare una base di W_1, W_2, $W_1 \cap W_2$, W_1+W_2 e le rispettive dimensioni:

~~$W_1 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : x + 2y + z - t = 0 \right\}$~~

$$W_1 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; 2x+y+z=0 \text{ e } y-z=0 \right\}$$

$$W_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; 2x+y=0 \text{ e } y=0 \right\}$$

Base per W_1:

$$\begin{cases} z = y \longrightarrow z = -x \\ 2x + y + y = 0 \longrightarrow y = -x \end{cases}$$

$x = x$
$y = -x$
$z = -x$
$t = t$

p.e $x=1$ $t=0$

$$W_1 = \left\langle \overset{\bar{v}_1}{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}} \overset{\bar{v}_2}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\rangle$$

$\dim W_1 = 2$

Base per W_2:

$$\begin{cases} 2x+y=0 \iff x=0 \\ y=0 \end{cases}$$

$x = 0$
$y = 0$
$z = z$
$t = t$

$z=1$, $t=0$ $z=0$, $t=1$

$$W_2 = \left\langle \overset{\bar{u}_1}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \overset{\bar{u}_2}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\rangle$$

$\dim W_2 = 2$

$W_1 \cap W_2$

$$\alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 = \beta_1 \bar{u}_1 + \beta_2 \bar{u}_2$$

$$\begin{cases} \alpha_1 = 0 \longrightarrow \alpha_1 = 0 \\ -\alpha_1 = 0 \\ -\alpha_1 = \beta_1 \longrightarrow \beta_1 = 0 \\ \alpha_2 = \beta_2 \end{cases}$$

per. $\alpha_2 = 1 = \beta_2$

$$W_1 \cap W_2 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$\Rightarrow \dim W_1 \cap W_2 = 1$

Base di $W_1 + W_2$:

$$W_1 + W_2 = \left\langle \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$\cancel{\begin{cases} a = 0 \\ -a = 0 \\ -a + c = 0 \\ b = 0 \end{cases}} \rightarrow \cancel{\begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}}$$

$$\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$$

$$= 2 + 2 - 1 = \boxed{3}$$

<u>Ex 6</u> | Determinare una base di W_1, W_2, $W_1 \cap W_2$, $W_1 + W_2$ e le rispettive dimensioni

$$W_1 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} : x + 2y + z - t = 0 \right\}$$

$$W_2 = \left\langle \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \\ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \end{pmatrix} & \begin{pmatrix} 5 \\ 0 \\ 4 \\ 4 \end{pmatrix} \end{array} \right\rangle$$

ⓌⒶ W_1

$$\begin{cases} x = -2y - z + t \\ y = y \\ z = z \\ t = t \end{cases}$$

p.e. $y = 1,\ z = 0,\ t = 0$
$y = 0,\ z = 1,\ t = 0$
$y = 0,\ z = 0,\ t = 1$

$$W_1 = \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

(W_2) Partiamo da un vettore, e lo aggiungiamo alla base se questo è lin. dip. rispetto ai vettori già presenti, in caso contrario lo scartiamo

- w_1 Partenza
- w_2?

w_1 e w_2 $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ sono lin. dip.

$\Rightarrow$ non sono multipli uno dell'altro.

· $w_3 = ?$

$$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \bar{0}$$

$$\begin{cases} a + b + c = 0 \iff a = 0 \\ 0 = 0 \\ b + c = 0 \iff c = 0 \\ 2b = 0 \iff b = 0 \end{cases}$$

✓ Lin. Dip.

$\Rightarrow$ Agg.

· $w_4 = ?$

$$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + d\begin{pmatrix}3\\0\\2\\3\end{pmatrix} = \bar{0}$$

$$\begin{cases} a + b + c + 3d = 0 \\ 0 = 0 \\ b + c + 2d = 0 \\ 2b + 3d = 0 \end{cases}$$

$a = -3\left(\frac{2b}{3}\right) - b - c$

$\cancel{b}\ c = -2\left(-\frac{2b}{3}\right) - b$

$d = -\frac{2b}{3}$

$\Rightarrow$ ~~sono $\neq 0$~~ posso essere $\neq 0$

$\Rightarrow$ NO L.D.

$\Downarrow$

SCARTIAMO W_4

· $W_5 = ?$

$$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + d\begin{pmatrix}5\\0\\4\\4\end{pmatrix}$$

$$\begin{cases} a + b + c + 5d = 0 \\ 0 = 0 \\ b + c + 4d = 0 \\ 2b + 4d = 0 \end{cases}$$

simile

$\Downarrow$

possono essere $\neq 0$

$\Rightarrow$ NO L.D

$\Downarrow$

scartiamo w_5

Base $(W_2) = \{w_1, w_2, w_3)$ $\boxed{\dim W_2 = 3}$

$W_1 \cap W_2$

~~Grassmann~~

~~$\dim(W_1+W_2) = \dim W_1 + \dim W_2$~~

$$w \in W_2 \Leftrightarrow w = a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}a+b+c\\0\\b+c\\2b\end{pmatrix} \qquad a,b,c \in \mathbb{R}$$

$$w \in W_1 \Leftrightarrow \quad W_1 : \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x+2y+z-t=0\right\}$$

$$w \in W_1 \Leftrightarrow \quad (a+\cancel{b}+c)+\cancel{20}+\cancel{b}+c-\cancel{2b}=0$$

$$\boxed{a=-2c}$$

$$w \in W_1 \cap W_2 \qquad w = \begin{pmatrix}b-c\\0\\b+c\\2b\end{pmatrix} = b\begin{pmatrix}1\\0\\1\\2\end{pmatrix} + c\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$$

$$W_1 \cap W_2 = \left\langle \begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix} \right\rangle \leftarrow \text{Base} \qquad \dim 2$$

$W_1 + W_2$

$$W_1 + W_2 = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \right\rangle$$

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

$$\underset{2}{\mid} \quad 3 + 3 - 2 = \boxed{4}$$

$$W_1 + W_2 = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\0\\0\end{pmatrix} \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\rangle$$

Base V