Soluzioni tutorato

1|

$$A+B = \begin{pmatrix} 3+9 & 5+6 & 8+0 \\ 9-9 & 10+10 & -9+9 \\ 2+6 & 9+0 & -9-8 \end{pmatrix} = \begin{pmatrix} 12 & 11 & 8 \\ \emptyset & 20 & \emptyset \\ 8 & 9 & -17 \end{pmatrix}$$

$$(A+B) * 6 = \begin{pmatrix} 72 & 66 & 48 \\ \emptyset & 120 & \emptyset \\ 48 & 54 & -102 \end{pmatrix}$$

2| $W = \{(x,y) \mid x,y \in \mathbb{R}^2, x+y=0\}$

(i*) $\bar{0} \in W$? $\quad 0+0=0$ ✓

(i) $\bar{u} \in W \quad \lambda \in \mathbb{R} \quad \lambda\bar{u} \in W$?

$\bar{u} = (x_1, y_1) \quad \lambda\bar{u} = (\lambda x_1, \lambda y_1) \quad \Rightarrow \lambda x_1 + \lambda y_1 = 0$?

$\lambda(\underbrace{x_1+y_1}_{=0}) = 0$ ✓

(ii) $\bar{u}, \bar{v} \in W \quad \bar{u}+\bar{v} \in W$?

$\bar{u} = (x_1, y_1) \quad \bar{v} = (x_2, y_2) \quad \bar{u}+\bar{v} = (x_1+x_2, y_1+y_2)$

$(x_1+x_2) + (y_1+y_2) = 0$ ✓

$0 \qquad 0$

3| $W = \{(x,y) \mid x,y \in \mathbb{R}^2\ xy=0\}$

i^* $\bar{0} \in W$? $\quad 0 \cdot 0 = 0$ ✓

(i) $\bar{u} \in W \quad \lambda \in \mathbb{R} \quad \lambda\bar{u} \in W$? $\quad \lambda x_1 \cdot \lambda y_1 = 0$

$\lambda^2(\underbrace{x_1 y_1}_{=0}) = 0$ ✓

(ii) $\bar{u}, \bar{v} \in W \quad \bar{u}+\bar{v} \in W$

$(x_1+x_2)(y_1+y_2) = \overset{0}{x_1y_1} + \underbrace{x_1y_2 + x_2y_1}_{\text{Non è detto che siano negativi}} + \overset{0}{x_2y_2}$

NON È SOTTOSP. $\Leftarrow$ Non è detto che siano negativi

Ex. 4.

$\mathbb{R}^2$

$\begin{pmatrix} x \\ y \end{pmatrix} = u \in \mathbb{R}^2 \quad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 v + \lambda_2 w$$
$$= \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{cases} x = \lambda_1 - \lambda_2 \\ y = 2\lambda_1 + \lambda_2 \end{cases} \quad \begin{aligned} \lambda_2 &= \lambda_1 - x \\ y &= 2\lambda_1 + \lambda_1 - x \end{aligned}$$

$$\lambda_1 = \frac{x+y}{3}$$

$$\lambda_2 = \frac{-2x+y}{3}$$

$$u = \begin{pmatrix} x \\ y \end{pmatrix} = \alpha_1 v + \alpha_2 w$$

$$u = \beta_1 v + \beta_2 w$$

$$\alpha_1 v + \alpha_2 w = \beta_1 v + \beta_2 w$$

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{cases} \alpha_1 - \alpha_2 = \beta_1 - \beta_2 \\ 2\alpha_1 + \alpha_2 = 2\beta_1 + \beta_2 \end{cases}$$

$$\begin{cases} \alpha_1 = \alpha_2 + \beta_1 - \beta_2 \\ 2\alpha_2 + \cancel{2\beta_1} - 2\beta_2 + \alpha_2 = \cancel{2\beta_1} + \beta_2 \end{cases}$$

$$\begin{cases} 3\alpha_2 = 3\beta_2 \longrightarrow \alpha_2 = \beta_2 \\ \alpha_1 = \beta_1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = \beta_2 \end{cases} \Longrightarrow$$ si scrive in modo unico come comb. lin. di v e w.

5) $w = \{(x,y,z) \mid x,y,z \in \mathbb{R}^3, x^2+y^2=z\}$

i^* $\bar{0} \in W$? $\quad 0+0=0$ ✓

(i) $\lambda\bar{u} = (\lambda x_1, \lambda y_1, \lambda z_1)$

$$(\lambda x_1)^2 + (\lambda y_1)^2 = \lambda z$$
$$\lambda^2 x_1^2 + \lambda^2 y_1^2 = \lambda z$$
$$\lambda^2 (x_1^2 + y_1^2) = \lambda z$$
$$\lambda (x_1^2 + y_1^2) = z$$

✗ NON VALIDO PER TUTTI I VETTORI

=> NON È UN SOTTOSPAZ.

6) $w = \{(x,y,z) \mid x+y-4z=0\}$

i^* $\bar{0} \in W$? $\quad 0+0+0=0$ ✓

(i) $\lambda\bar{u} = (\lambda x_1, \lambda y_1, \lambda z_1)$

$$\lambda x_1 + \lambda y_1 - 4\lambda z_1 = 0$$
$$\lambda(\underbrace{x_1 + y_1 - 4z_1}_{=0}) = 0 \quad ✓$$

(ii) $\bar{u}+\bar{v} = (x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$$(x_1+x_2) + (y_1+y_2) + (z_1+z_2) = 0 \quad ✓$$

$x+y-4z=0$

$$x = 4z - y$$
$$y = y$$
$$z = z$$

$$\begin{pmatrix} 4z-y \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$U = \left\langle \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

Ex8

(a) $\mathbb{R}^4$

$$V = \left\langle \underset{v_1}{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}} \right\rangle \quad W = \left\langle \underset{w_1}{\begin{pmatrix}0\\1\\0\\1\end{pmatrix}}, \underset{w_2}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}} \right\rangle$$

$$V+W = \left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\rangle$$

$$w_1 = v_1 + v_2 - w_2$$

$$= \langle v_1, v_2, w_2 \rangle$$

$$V \cap W = ?$$

$$\alpha_1 v_1 + \alpha_2 v_2 = \beta_1 w_1 + \beta_2 w_2$$

$$\alpha_1 \begin{pmatrix}1\\0\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \beta_1 \begin{pmatrix}0\\1\\0\\1\end{pmatrix} + \beta_2 \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\begin{cases} \alpha_1 = \beta_2 \\ \alpha_2 = \beta_1 \\ 0 = 0 \\ \alpha_1 = \beta_1 \end{cases} \qquad \begin{cases} \alpha_1 = \beta_1 \\ \alpha_2 = \beta_1 \\ \beta_2 = \beta_1 \\ \beta_1 \text{ qualsiasi} \end{cases}$$

$$V \cap W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

ESERCIZIO 2 Considerate

$$S = \left\langle \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} \right\rangle \qquad T = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\rangle$$

(a) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \in S$?

(b) Trovare generatori per $S \cap T$, $S+T$.

(c) È vero che $S+T = \mathbb{R}^3$?

(a) $v \in S \iff v$ può essere scritto come combinazione lineare di $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ e $\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$

$\iff \exists\, a, b \in \mathbb{R} \quad v = a \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + b \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$.

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + b \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ 4b \\ -3a-b \end{pmatrix} \qquad \begin{pmatrix} a \\ 4b \\ -3a-b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} a = 2 \\ 4b = 1 \\ -3a-b = 1 \end{cases} \qquad \begin{matrix} a = 2 \\ b = 1/4 \\ -6 - \frac{1}{4} \neq 1 \end{matrix} \qquad \text{il sistema non ha soluzioni}$$

$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \notin S$.

(b) Generatori di $S+T$ $\qquad S+T = \left\langle \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\rangle$

Dati un insieme di generatori di S e un insieme di generatori di T, allora l'unione

Generatori di $S \cap T$

$v \in S \cap T \iff v \in S$ e $v \in T$

dei due insiemi è un insieme di generatori per $S+T$. (vero per ogni S,T).

$$v \in S \iff v = a\begin{pmatrix}1\\0\\-3\end{pmatrix} + b\begin{pmatrix}0\\4\\-1\end{pmatrix} = \begin{pmatrix}a\\4b\\-3a-b\end{pmatrix} \quad (\star)$$

$$v \in T \iff v = c\begin{pmatrix}1\\0\\0\end{pmatrix} + d\begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}c+2d\\-d\\3d\end{pmatrix} \quad (\#)$$

$$v \in S\cap T \iff \begin{pmatrix}a\\4b\\-3a-b\end{pmatrix} = \begin{pmatrix}c+2d\\-d\\3d\end{pmatrix}$$ capiamo quando ciò è possibile.

$$\begin{cases} a = c+2d \\ 4b = -d \\ -3a-b = 3d \end{cases} : \quad \begin{cases} a \quad -c-2d = 0 \\ 4b \quad +d = 0 \\ -3a-b \quad -3d = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 4 & 0 & 1 \\ -3 & -1 & 0 & -3 \end{pmatrix} \xrightarrow{III+3I} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 4 & 0 & 1 \\ 0 & -1 & -3 & -9 \end{pmatrix}$$

$$\begin{array}{c} \\ -III \\ II \end{array}\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 9 \\ 0 & 4 & 0 & 1 \end{pmatrix} \xrightarrow{III-4II} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & -12 & -35 \end{pmatrix}$$

R-C: l'insieme delle soluzioni dipende da un parametro.

$$\begin{cases} a \quad -c-2d = 0 \\ b + 3c + 9d = 0 \\ -12c - 35d = 0 \end{cases} \qquad d = s$$

$$c = -\frac{35}{12}s$$

$$b = -3c - 9d = \frac{35}{4}s - 9s = -\frac{s}{4}$$

$$a = c + 2d = -\frac{35}{12}s + 2s = -\frac{11}{12}s$$

$$\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}-\frac{11}{12}s\\-\frac{1}{4}s\\-\frac{35}{12}s\\s\end{pmatrix} = s\begin{pmatrix}-11/12\\-1/4\\-35/12\\1\end{pmatrix} \quad s \in \mathbb{R}$$

$$v = \begin{pmatrix}a\\4b\\-3a-b\end{pmatrix} = \begin{pmatrix}-\frac{11}{12}s\\-s\\3s\end{pmatrix} \qquad -3a-b = \frac{11}{4}s + \frac{s}{4} = \frac{12}{4}s = 3s$$

$$= s\begin{pmatrix}-11/12\\-1\\3\end{pmatrix}$$

$$S \cap T = \left\langle \begin{pmatrix}-11/12\\-1\\3\end{pmatrix} \right\rangle$$

(d) $S+T = \left\langle \begin{pmatrix}1\\0\\-3\end{pmatrix}, \begin{pmatrix}0\\4\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-1\\3\end{pmatrix} \right\rangle \stackrel{?}{=} \mathbb{R}^3$

$v \in \mathbb{R}^3 \quad v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad v \in S+T \iff$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + b \begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \text{per qualche } a,b,c,d \in \mathbb{R}.$$

Vogliamo capire per quali $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, il sistema

$$\begin{cases} a + c + 2d = x \\ 4b - d = y \\ -3a - b + 3d = z \end{cases}$$

ha soluzione. (Le incognite sono a, b, c, d).

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & x \\ 0 & 4 & 0 & -1 & y \\ -3 & -1 & 0 & 3 & z \end{array}\right) \qquad \underset{III+3I}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & x \\ 0 & 4 & 0 & -1 & y \\ 0 & -1 & 3 & 9 & z+3x \end{array}\right)$$

$$\underset{II \leftrightarrow III}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & x \\ 0 & -1 & 3 & 9 & z+3x \\ 0 & 4 & 0 & -1 & y \end{array}\right) \qquad \underset{III+4II}{\longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & x \\ 0 & -1 & 3 & 9 & z+3x \\ 0 & 0 & 12 & -35 & y+4z+12x \end{array}\right)$$

R-C : non c'è pivot nella colonna dei termini noti, quindi il sistema ha soluzione (qualsiasi sia il valore di x, y, z).

Quindi $S+T = \mathbb{R}^3$.