SISTEMI LINEARI

Def). Un SIST. LIN. di m equazioni in n incognite, a coeff. reali è una lista di m equazioni in $x_1, ~~x_2~~, \ldots x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{21}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

se $b_1 = b_2 = \ldots = b_m = 0 \Rightarrow$ OMOGENEO!

Definiamo

$A_{m \times n}$ Matrice incompleta $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

• Colonna di termini noti $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

• Matrice completa $(A|\underline{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$

Matrice a "gradini" se $\forall$ riga il primo elemento non nullo è più a dx del primo elemento non nullo sulla riga precedente

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \boxed{a_{22}} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & \boxed{a_{mn}} \end{pmatrix}$$

PIVOT

↳ è importante perché se è a gradini il SL si risolve per sostituzione partendo dalla variabile pivot più in basso

se non è in scala?

3 Operazioni

(i) moltiplicare un'eq. per uno scalare non nullo

(ii) riordinare le equaz.

(iii) sommare ad un equaz. del sistema il multiplo di un'altra equaz.

↳ Le applichiamo alle righe della mat completa

=> ALGORITMO DI GAUSS

~~partiamo una matrice~~
ci permette di ottenere una matrice a GRADINI

TEOREMA DI ROUCHE'-CAPELLI: sia

Dato un S.L ~~e~~ non omogeneo, ~~tale~~ $(A|\underline{b})$ la matrice completa: attraverso l'alg. di Gauss otteniamo una MAT a gradini.
Il S.L ammette sol. $\Leftrightarrow$ non c'è pivot sulla colonna dei termini noti.
Inoltre, se il sistema ammette soluzioni, queste saranno descritte da tanti parametri quante sono le variabili NON-PIVOT.

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SISTEMI LINEARI.

Ex. 1.

$$\begin{cases} 2x_1 - 4x_2 = -4 \\ 3x_1 - 6x_2 + 3x_3 = -3 \\ x_1 - 2x_2 - x_3 = -2 \end{cases}$$

$$A|b = \left(\begin{array}{ccc|c} 2 & -4 & 0 & -4 \\ 3 & -6 & 3 & -3 \\ 1 & -2 & -1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 3 & -6 & 3 & -3 \\ 1 & -2 & -1 & -2 \end{array}\right) \quad I/2$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \mathrm{I} \\ \mathrm{II} - 3\mathrm{I} \\ \mathrm{III} - \mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right) \begin{array}{l} \mathrm{I} \\ \mathrm{III} \\ \mathrm{II} + 3\mathrm{III} \end{array}$$

$0 = 3$ IMPOSS. $\Downarrow$

$\Longrightarrow$ NO SOLUZIONE!

Ex. 2.

$$\begin{cases} 2x_1 - 2x_2 + 8x_3 = 5 \\ 2x_2 + 6x_3 = 1 \\ x_1 - 2x_2 + 4x_3 = -1 \\ x_1 + 10x_3 = 0 \end{cases} \qquad A|b$$

$$A|b = \left(\begin{array}{ccc|c} 2 & -2 & 8 & 5 \\ 0 & 2 & 6 & 1 \\ 1 & -2 & 4 & -1 \\ 1 & 0 & 10 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 2 & 6 & 1 \\ 2 & -2 & 8 & 5 \\ 1 & 0 & 10 & 0 \end{array}\right)\begin{array}{l} \mathrm{III} \\ \mathrm{II} \\ \mathrm{I} \\ \mathrm{IV} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 2 & 6 & 1 \\ 0 & 2 & 0 & 7 \\ 0 & 2 & 6 & 1 \end{array}\right)\begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}-2\mathrm{I} \\ \mathrm{IV}-\mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 1 & 3 & 1/2 \\ 0 & 2 & 0 & 7 \\ 0 & 2 & 6 & 1 \end{array}\right)\begin{array}{l} \mathrm{I} \\ \mathrm{II}/2 \\ \mathrm{III} \\ \mathrm{IV} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 1 & 3 & 1/2 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l} \mathrm{I} \\ \mathrm{II}/2 \\ \mathrm{III}-2\mathrm{II} \\ \mathrm{IV}-2\mathrm{II} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 1 & 3 & 1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)\begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}/-6 \\ \mathrm{IV} \end{array}$$

$$\begin{cases} x_1 - 2x_2 + 4x_3 = -1 \\ x_2 + 3x_3 = \frac{1}{2} \\ x_3 = -1 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 = 3 \rightarrow x_1 = 3 + 7 = 10 \\ x_2 = \frac{1}{2} + 3 = 7/2 \\ x_3 = -1 \end{cases}$$

$$S = \left\{ x^* = \begin{pmatrix} 10 \\ 7/2 \\ -1 \end{pmatrix} \right\}.$$

Ex. 3.

$$\begin{cases} 2x_1 + 5x_3 = 1 \\ 4x_1 - 3x_2 + 4x_3 = 5 \\ 2x_1 - x_2 + 3x_3 = 2 \end{cases}$$

$$A|b = \left(\begin{array}{ccc|c} 2 & 0 & 5 & 1 \\ 4 & -3 & 4 & 5 \\ 2 & -1 & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5/2 & 1/2 \\ 4 & -3 & 4 & 5 \\ 2 & -1 & 3 & 2 \end{array}\right) \begin{array}{l} \mathrm{I}/2 \\ \mathrm{II} \\ \mathrm{III} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5/2 & 1/2 \\ 0 & -3 & -6 & 3 \\ 0 & -1 & -2 & 1 \end{array}\right) \begin{array}{l} \mathrm{I} \\ \mathrm{II}-4\mathrm{I} \\ \mathrm{III}-2\mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5/2 & 1/2 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ 3\mathrm{III}-\mathrm{II} \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5/2 & 1/2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \mathrm{I} \\ \mathrm{II}/-3 \\ \mathrm{III} \end{array}$$

$$\begin{cases} x_1 + \frac{5}{2} x_3 = \frac{1}{2} \\ x_2 + 2x_3 = -1 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x_1 = -\frac{5}{2} x_3 + \frac{1}{2} \\ x_2 = -2 x_3 - 1 \\ x_3 \text{ qualsiasi} \end{cases}$$

$$S = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} x_3 + \frac{1}{2} \\ -2x_3 - 1 \\ x_3 \end{pmatrix}, \; x_3 \in \mathbb{R} \right\}.$$

Ex. 4.

$m \in \mathbb{R}$

$$\begin{cases} mx + (m-1)y = m+2 \\ (m+1)x - my = 5m+3 \end{cases}$$

$$A|b = \left(\begin{array}{cc|c} m & m-1 & m+2 \\ m+1 & -m & 5m+3 \end{array}\right)$$

- $m=0$

$$\left(\begin{array}{cc|c} 0 & -1 & 2 \\ 1 & 0 & 3 \end{array}\right) \longrightarrow \begin{cases} x=3 \\ y=-2 \end{cases}$$

- $m \neq 0$

$$\left(\begin{array}{cc|c} m & m-1 & m+2 \\ m+1 & -m & 5m+3 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & \frac{m-1}{m} & \frac{m+2}{m} \\ m+1 & -m & 5m+3 \end{array}\right) \begin{array}{l} \frac{I}{m} \\ II \end{array}$$

$$\left(\begin{array}{cc|c} 1 & \frac{m-1}{m} & \frac{m+2}{m} \\ 0 & \frac{1-2m^2}{m} & \frac{4m^2-2}{m} \end{array}\right) \begin{array}{l} I \\ II-(m+1)I \end{array}$$

$$-m-(m+1)\frac{m-1}{m} = -m-\frac{m^2-1}{m} =$$

$$= \frac{1-2m^2}{m}$$

$$5m+3-(m+1)\frac{(m+2)}{m} =$$

$$= \frac{5m^2+\cancel{3m}-m^2-\cancel{3m}-2}{m} = \frac{4m^2-2}{m}$$

- $m^2 = \frac{1}{2} \Longrightarrow m = \pm 1/\sqrt{2}$

$$\left(\begin{array}{cc|c} 1 & \frac{m-1}{m} & \frac{m+2}{m} \\ \hline 0 & 0 & 0 \end{array}\right)$$

INFINITE SOLUZIONI

$$\begin{cases} x = \frac{1-m}{m}\, y + \frac{m+2}{m} \\ y = \text{qualsiasi} \end{cases}$$

- $m^2 \neq \frac{1}{2}$

$$\left(\begin{array}{cc|c} 1 & \frac{m-1}{m} & \frac{m+2}{m} \\ 0 & \frac{1-2m^2}{m} & \frac{4m^2-2}{m} \end{array}\right)$$

$$y = \frac{4m^2-2}{\cancel{m}} \cdot \frac{\cancel{m}}{1-2m^2} = \frac{4m^2-2}{1-2m^2} = -2$$

$$x = -\frac{m-1}{m} y + \frac{m+2}{m} = -\frac{m-1}{m} \cdot \underbrace{\frac{4m^2-2}{1-2m^2}}_{=-2} + \frac{m+2}{m} = \frac{2m-\cancel{2}+m\cancel{+2}}{m} = \frac{3\cancel{m}}{\cancel{m}} = 3$$

$$\begin{cases} x = 3 \\ y = -2 \end{cases}$$ UNICA SOL.

EX. 5.

$\lambda \in \mathbb{R}$

$$\begin{cases} x+2y+2w+z=1 \\ y+2w+z=0 \\ x+y+\lambda w=0 \\ \lambda y+2\lambda w+\lambda^2 z=0 \end{cases}$$

$$A|b = \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & \lambda & 0 & 0 \\ 0 & \lambda & 2\lambda & \lambda^2 & 0 \end{array}\right)$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -1 & \lambda-2 & -1 & -1 \\ 0 & \lambda & 2\lambda & \lambda^2 & 0 \end{array}\right) \end{array} \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}-\mathrm{I} \\ \mathrm{IV} \end{array}$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \lambda^2-\lambda & 0 \end{array}\right) \end{array} \begin{array}{l} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}+\mathrm{II} \\ \mathrm{IV}-\lambda\,\mathrm{II} \end{array}$$

- $\lambda = 0$

↳ NO Sol.

• $\lambda = 1$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x + 2y + 2w + z = 1 \\ y + 2w + z = 0 \\ w = -1 \\ 0 = 0 \end{cases}$$

$$\begin{cases} x + 2y + z = 3 \\ y + z = 2 \\ w = -1 \end{cases}$$

$$\begin{cases} w = -1 \\ y = 2 - z \\ x = 3 - z - 2y = 3 - z - 4 + 2z \\ \qquad\qquad\quad = z - 1 \\ z \text{ qualsiasi} \end{cases}$$

INFINITE SOL.

$$S = \left\{ \begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix} = \begin{pmatrix} z-1 \\ 2-z \\ -1 \\ z \end{pmatrix},\ z \in \mathbb{R} \right\}$$

• $\lambda \neq 0, 1$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \underbrace{\lambda^2-\lambda}_{\lambda(\lambda-1)} & 0 \end{array}\right)$$

$z(\lambda^2-\lambda) = 0 \longrightarrow z = 0$

$\lambda w = -1 \longrightarrow w = -1/\lambda$

$y = -2w - z = \dfrac{2}{\lambda}$

$x = -2y - 2w - z + 1$

$= -\dfrac{4}{\lambda} + \dfrac{2}{\lambda} + 1 = -\dfrac{2}{\lambda} + 1.$