2020/05/26

martedì 26 maggio 2020 13:57

F9/E2

$\phi: \mathbb{R}^3 \to \mathbb{R}^3$ t.c. $\phi\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$\phi\begin{pmatrix}0\\1\\0\end{pmatrix} = \phi\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

(a) $A_{\mathcal{E}\mathcal{E}}(\phi) = ?$ $\phi\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = ?$

(b) $\phi^{-1}\begin{pmatrix}2\\2\\1\end{pmatrix} = ?$ se è sp. vett.?

(c) ϕ è diag.

(a)

$$A_{\mathcal{E}\mathcal{E}}(\phi) = \begin{pmatrix}1 & 1 & 1\\1 & 1 & 1\\0 & 1 & 1\end{pmatrix} =: A$$

$$\phi\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = A\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1 + x_2 + x_3\\x_1 + x_2 + x_3\\x_2 + x_3\end{pmatrix}$$

(b)

$$\phi^{-1}\begin{pmatrix}2\\2\\1\end{pmatrix} = \left\{ v \in \mathbb{R}^3 \mid \phi(v) = \begin{pmatrix}2\\2\\1\end{pmatrix} \right\}$$

$$= \left\{ v \in \mathbb{R}^3 \mid Av = \begin{pmatrix}2\\2\\1\end{pmatrix} \right\}$$

$Ax = b$

$$= v_0 + \mathrm{Ker}(A) = \begin{pmatrix}1\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle$$

v_0: soluzione particolare

$\begin{pmatrix}1\\1\\0\end{pmatrix} = v_0$

$$\begin{pmatrix}1&1&1\\1&1&1\\0&1&1\end{pmatrix}\begin{pmatrix}x\\y\\0\end{pmatrix} = \begin{pmatrix}2\\2\\1\end{pmatrix}$$

$$x\begin{pmatrix}1\\1\\0\end{pmatrix} + y\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}x+y\\x+y\\y\end{pmatrix}$$

$$\begin{cases} x = 1 \\ y = 1 \end{cases}$$

$$\operatorname{Im} A = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle \Rightarrow rk A = 2$$

$$\Rightarrow \dim \operatorname{Ker}(A) = 3-2 = 1$$

$\phi^{-1}\begin{pmatrix}2\\2\\1\end{pmatrix}$ NON è sp. vett. perché $\begin{pmatrix}1\\1\\0\end{pmatrix} \in \phi^{-1}\begin{pmatrix}2\\2\\1\end{pmatrix}$

$$\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix} \notin \phi^{-1}\begin{pmatrix}2\\2\\1\end{pmatrix}$$

V è sp. vet. $\Leftrightarrow$ $v+w \in V \quad \forall v,w \in V$

$\lambda v \in V \quad \forall v \in V, \lambda \in \mathbb{R}$

③ $$\det(x\mathbb{1} - A) = \begin{vmatrix} x-1 & -1 & -1 \\ -1 & x-1 & -1 \\ 0 & -1 & x-1 \end{vmatrix} = (x-1)\left[x^2 - 2x\right] - (-1)\left(-x+1-1\right)$$

$$= x(x-1)(x-2) - x$$

$$= x\left(x^2 - 3x + 1\right)$$

$$= x\left(x - \frac{3+\sqrt{5}}{2}\right)\left(x - \frac{3-\sqrt{5}}{2}\right)$$

$$x_{1/2} = \frac{3 \pm \sqrt{9-4}}{2} \quad \begin{matrix} \frac{3+\sqrt{5}}{2} \\ \frac{3-\sqrt{5}}{2} \end{matrix}$$

m.a.: 1   1   1

$1 \leq$ m.g. $\leq$ m.a. $= 1 \Rightarrow$ m.g. $= 1 \Rightarrow \phi$ è diag.

F9/E3

• $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ $\begin{vmatrix} x-1 & -3 \\ -3 & x-1 \end{vmatrix} = (x-4)(x+2)$ ✓ diag.

• $B = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$ $\begin{vmatrix} x-1 & 3 \\ -3 & x-1 \end{vmatrix} = x^2 - 2x + 10$ $\Delta = 4 - 40 = -36 < 0$

$\Rightarrow \nexists$ radici reali $\Rightarrow$ ✗ no diag

• $C = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ $P_C(x) = (x-3)^2$ m.a. $= 2$

$C - 3\mathbb{1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $Ker(C - 3\mathbb{1}) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$

$\Rightarrow$ m.g. $= 1$ ✗

• $D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ ✓ è già diag.

• $E = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ $P_E(x) = \begin{vmatrix} x-1 & -2 & -5 \\ 0 & x & -3 \\ 0 & 0 & x-1 \end{vmatrix} = x(x-1)^2$

0 m.a. $= 1 \Rightarrow$ m.g. $= 1$

1 m.a. $= 2$

$\lambda = 1$ $E - \mathbb{1} = \begin{pmatrix} 0 & 2 & 5 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ $Ker(E - \mathbb{1}) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \Rightarrow$ m.g. $= 1$

- $F = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad P_F(x) = \begin{vmatrix} x-1 & -2 & -4 \\ 0 & x & 2 \\ 0 & 0 & x-1 \end{vmatrix} = x(x-1)^2$

$0 \quad m.a.=1 \Rightarrow m.g.=1$

$1 \quad m.a.=2$

$$F - \mathbb{1} = \begin{array}{c} \begin{pmatrix} 0 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} v_0 & v_1 & v_2 \end{array} \end{array} \qquad Ker(F-\mathbb{1}) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\rangle \quad m.g.=2 \Rightarrow \checkmark$$

$v_2 = 2v_1 \Rightarrow v_2 - 2v_1 = \overline{0}$

- $G = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ $\qquad P_G(x) = \begin{vmatrix} x-1 & 1 & -1 \\ 1 & x-1 & 1 \\ -1 & 1 & x-1 \end{vmatrix}$

$$= (x-1)(x^2-2x+\cancel{1}-\cancel{1}) - (x\cancel{-1}\cancel{+1}) - 1(\cancel{1}+x\cancel{-1})$$

$$= x^3-2x^2-x^2\cancel{+2x}\cancel{-x}\cancel{-x}$$

$$= x^2(x-3)$$

$3 \quad m.a.=1 \Rightarrow m.g.=1$

$0 \quad m.a.=2$

$$\mathrm{Ker}(G - 0\mathbb{1}) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \Rightarrow m.g. = 2$$ ✓

F10/E2

$$U_a = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1+a \\ -a \\ a-1 \end{pmatrix} \right\rangle \quad \text{in } \mathbb{R}^3 \qquad v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

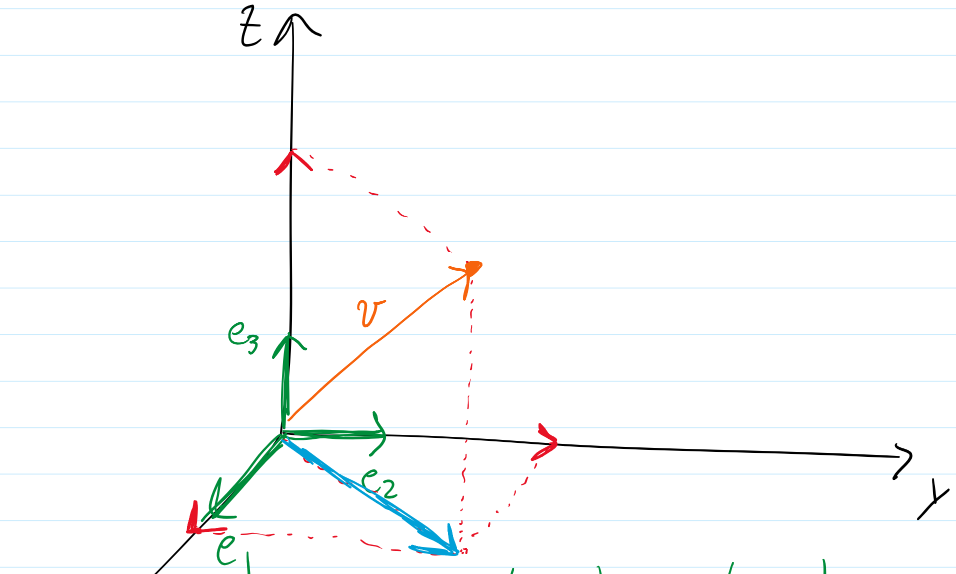
(a) $a=1$ proiezione $\perp$ di v su U_1 ?

(b) $a=1$ trovare $w \in \mathbb{R}^3$ con la stessa proiez. $\perp$ di v?

(c) a t.c. proiez. $\perp$ di v è nulla ?

(d) $a \in \mathbb{R}$ t.c. norma della proiezione $\perp$ di v su U_a è $\sqrt{13}$?

(a) $a=1 \Rightarrow U_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\rangle$



$$P(v) = (v \cdot e_1)e_1 + (v \cdot e_2)e_2$$
$$= v - (v \cdot e_3)e_3$$

X

$= v - (v \cdot e_3) e_3$

$$U_1^\perp : \begin{cases} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \end{cases} \iff \begin{cases} x + z = 0 \\ 2x - y = 0 \end{cases} \Rightarrow U_1^\perp = \left\langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\rangle \quad \|w\|^2 = 1+4+1 = 6$$

w

$$P_{U_1}(v) = v - \left(\frac{v \cdot w}{\|w\|^2} \right) w$$

$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \left(\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \frac{1}{6} \right) \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \\ 5/2 \end{pmatrix}$$

(b) $\{ w \in \mathbb{R}^3 \mid P_{U_1}(w) = P_{U_1}(v) \}$

$\Updownarrow$

$P_{U_1}(w) - P_{U_1}(v) = 0$

$\Updownarrow$ → P_{U_1} è lineare $\quad P_{U_1}(av + bw) = a P_{U_1}(v) + b P_{U_1}(w)$

$P_{U_1}(w - v) = 0$

$\Updownarrow$ → def. di $Ker(P_{U_1})$

$w - v \in Ker(P_{U_1})$

$\Updownarrow$

$w \in v + Ker(P_{U_1}) = v + U_1^\perp$

$$P_{U_1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$v + u \longmapsto v$$

$$v \longmapsto v \qquad v \in U_1$$
$$u \longmapsto 0 \qquad u \in U_1^{\perp}$$

$$\Downarrow$$

$$\mathrm{Ker}(P_{U_1}) = U_1^{\perp}$$

$$\Rightarrow w \in \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\rangle$$

ⓒ a t.c. $P_{U_a}(v) = 0$?

$$U_a^{\perp}: \begin{cases} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1+a \\ -a \\ a-1 \end{pmatrix} = 0 \end{cases} \iff \begin{cases} x + z = 0 \\ (1+a)x - ay + (a-1)z = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x + z = 0 \\ z(-1 - a + a - 1) - ay = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x + z = 0 \\ -ay - 2z = 0 \end{cases}$$

$$\Rightarrow U_a^{\perp} = \left\langle \underbrace{\begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}}_{w_a} \right\rangle$$

modo 1: $$P_{U_a}(v) = \underbrace{\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}_{v} - \underbrace{\frac{1}{2a^2+4}}_{\frac{1}{\|w_a\|^2}} \left(\underbrace{\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}_{v} \cdot \underbrace{\begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}}_{w_a} \right) \underbrace{\begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}}_{w_a}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \frac{a+2}{2a^2+4} \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

modo 2 : $v \in \ker(P_{U_a}) = U_a^{\perp}$: $\begin{cases} x + z = 0 \\ -ay - 2z = 0 \end{cases}$

$$\Rightarrow \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \in \left\langle \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix} \right\rangle$$

v non soddisfa mai le eq. cartesiane

$$\begin{cases} x + z = 0 \\ \sim\sim\sim \end{cases} \qquad \begin{cases} 2 + 3 = 0 \\ \end{cases} \quad ⚡$$

$$\nexists\, a \in \mathbb{R}$$

(d) $P_{U_a}(v) = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - \dfrac{1}{2a^2+4} \left(\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix} \right) \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}$

(v; $\frac{1}{2a^2+4} \to \frac{1}{\|w_a\|^2}$; v, w_a, w_a)

$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \frac{a+2}{2a^2+4} \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}$$

$$= \frac{1}{4+2a^2} \begin{pmatrix} 5a^2+2a+8 \\ -2a^2+2a \\ 5a^2-2a+12 \end{pmatrix} =: u_a$$

$$\|u_a\| = \sqrt{13} \iff \sqrt{u_a \cdot u_a} = \sqrt{13} \iff u_a \cdot u_a = 13$$

$$u_a \cdot u_a = \frac{1}{(4+2a^2)^2}\left(54a^4 - 8a^3 + 212a^2 - 16a + 208\right) = 13$$

$$\Updownarrow$$

$$2a^4 - 8a^3 + 4a^2 - 16a = 0$$

$$\Updownarrow$$

$$2a^4-8a^3+4a^2-16a=0$$

$$\Updownarrow$$

$$2a\left(a^3-4a^2+2a-8\right)=0$$

$\swarrow$ $\searrow$

$a=0$ ✓

$$a^3-4a^2+2a-8=0$$

$$\Updownarrow$$

$$a^2(a-4)+2(a-4)=0$$

$$\Updownarrow$$

$$(a-4)(a^2+2)=0$$

$$\Updownarrow$$

$$a=4$$

Es extra

$\mathbb{R}^4$: $U=\left\langle\begin{pmatrix}\vdots\end{pmatrix}\right\rangle$ $\quad V=\left\{\begin{pmatrix}x\\y\\z\\t\end{pmatrix} \middle| \begin{cases}x+y+z+t=0\\2x+y-z-3t=0\end{cases}\right\}$

(a) $\dim V=?$ base di $V=?$

(b) eq. cart. di $U+V$? somma diretta ?

(c) base di $U+V$? completarla a $\mathbb{R}^4$

(a) $\left(\begin{array}{cccc|c}1&1&1&1&0\\2&1&-1&-3&0\end{array}\right)\xrightarrow{II-2I}\left(\begin{array}{cccc|c}1&1&1&1&0\\0&-1&-3&-5&0\end{array}\right)$

$\Rightarrow \begin{cases}x+y+z+t=0\\y+3z+5t=0\end{cases} \Rightarrow \begin{cases}x=3z+5t-z-t=2z+4t\\y=-3z-5t\end{cases}$

$$\Rightarrow \begin{cases} x+y+z+t=0 \\ y+3z+5t=0 \end{cases} \Rightarrow \begin{cases} x = 3z+5t-z-t = 2z+4t \\ y=-3z-5t \end{cases}$$

$$\Rightarrow V = \left\{ \begin{pmatrix} 2z+4t \\ -3z-5t \\ z \\ t \end{pmatrix} \;\middle|\; z,t \in \mathbb{R} \right\}$$

$$= \left\{ z \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; z,t \in \mathbb{R} \right\}$$

$$= \left\langle \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \dim V = 2$$

(b) $U \cap V = \langle 0 \rangle$ $\qquad V: \begin{cases} x+y+z+t=0 \\ 2x+y-z-3t=0 \end{cases}$ $\qquad x+y+z+t=0 \Leftrightarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0$

$\Rightarrow V \subseteq U^{\perp}$

$$U+V = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\rangle \mapsto \text{base di } U+V$$

$\qquad v_1 \qquad v_2 \qquad v_3$

$$\underset{U+V}{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}} = a \cdot v_1 + b\, v_2 + c\, v_3 \Rightarrow \begin{cases} x = a+2b+4c \\ y = a-3b-5c \\ z = a+b \\ t = a+c \end{cases}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 4 \\ 1 & -3 & -5 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{vmatrix} x \\ y \\ z \\ t \end{vmatrix}}_{b} \xrightarrow{\begin{matrix} IV \\ II-IV \\ III-IV \\ I-IV \end{matrix}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & t \\ 0 & -3 & -6 & y-t \\ 0 & 1 & -1 & z-t \\ 0 & 2 & 3 & x-t \end{array} \right) \dots$$

$$\dots \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & t \\ 0 & 1 & -1 & z-t \\ \end{array} \right.$$

$$
\dashrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & t \\ 0 & 1 & -1 & z-t \\ 0 & 0 & -9 & y+3z-4t \\ 0 & 0 & 0 & 9x+5y-3z-11t \end{array}\right)
$$

$$9x+5y-3z-11t = 0$$

$$\Rightarrow U+V \left\{ 9x+5y-3z-11t=0 \right.$$

ⓒ $U+V = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$v_1 \quad v_2 \quad v_3$

$$\mathbb{R}^4 = \left\langle v_1 \quad v_2 \quad v_3 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$