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F8/E6

$$\phi_a : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad \text{t.c.} \quad \phi_a\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + ay \\ (1-a)y + z \\ ax + y + 2z \end{pmatrix}$$

a) $A_{\mathcal{E}\mathcal{E}}(\phi_a) = ? =: A_a$

b) trovare a t.c. ϕ_a NON sia suriettiva ?

c) per tali a $\ker = ?$ $\operatorname{Im} = ?$

d) $\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\}$, $\quad A_{\mathcal{A}\mathcal{A}}(\phi_a) = ?$

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a) $A_a \in M_3(\mathbb{R})$

$$A_a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \phi_a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} \mapsto 1^\circ \text{ colonna di } A$$

$$\phi_a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 1-a \\ 1 \end{pmatrix}$$

$$\phi_a\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$A_a = \begin{pmatrix} 1 & a & 0 \\ 0 & 1-a & 1 \\ a & 1 & 2 \end{pmatrix}$$

b) $0 = \det(A_a)$

$$= 1 \cdot \begin{vmatrix} 1-a & 1 \\ 1 & 2 \end{vmatrix} + a \begin{vmatrix} a & 0 \\ 1-a & 1 \end{vmatrix} = 2(1-a) - 1 + a(a)$$

$$= a^2 - 2a + 2 - 1 = (a-1)^2$$

$\Rightarrow \phi_a$ non è sur. $\Leftrightarrow a = 1$

c) $a = 1 \Rightarrow A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \Rightarrow \operatorname{Im} A_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cancel{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle \Rightarrow rk(A_1) = 2$$

$$\Rightarrow \dim(\mathrm{Ker}(A_1)) = \dim(\mathbb{R}^3) - \mathrm{rk}(A_1) = 3-2=1$$

$$\Rightarrow \mathrm{Ker}(A_1) = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

(d) $A_{\alpha\alpha}(\Phi_a) = A_{\varepsilon\alpha}(id)\, A_{\varepsilon\varepsilon}(\Phi_a)\, A_{\alpha\varepsilon}(id)$

$$\begin{tikzcd} \mathbb{R}^3_\varepsilon \arrow[r, "A_{\varepsilon\varepsilon}(\Phi_a)"] & \mathbb{R}^3_\varepsilon \arrow[d, "A_{\varepsilon\alpha}(id)"] \\ \mathbb{R}^3_\alpha \arrow[u, "A_{\alpha\varepsilon}(id)"] \arrow[r, "A_{\alpha\alpha}(\Phi_a)"'] & \mathbb{R}^3_\alpha \end{tikzcd}$$

$$A_{\alpha\varepsilon}(id) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix} =: P$$

$$A_{\varepsilon\alpha}(id) = \left(A_{\alpha\varepsilon}(id)\right)^{-1} = \frac{1}{-4}\begin{pmatrix} 2 & -4 & -2 \\ -3 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix}$$

$$|P| = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 2-2(2+1) = 2-6 = -4$$

$$\Rightarrow A_{\alpha\alpha}(\Phi_a) = \left(\frac{1}{-4}\begin{pmatrix} 2 & -4 & -2 \\ -3 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix}\right)\underbrace{\begin{pmatrix} 1 & a & 0 \\ 0 & 1-a & 1 \\ a & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}}$$

$$= -\frac{1}{4}\begin{pmatrix} 2 & -4 & -2 \\ -3 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} a+1 & 2+a & -a \\ 1-a & 1-a & 1+a \\ a+3 & 2a+1 & 3 \end{pmatrix}$$

$$= \boxed{-\frac{1}{4}\begin{pmatrix} 4a-12 & 2a-2 & -6a-10 \\ -4a+4 & -3a-3 & 5a+5 \\ -4a & -5a-1 & 3a-1 \end{pmatrix}}$$

$\boxed{F9/E1}$

$A \in M_2(\mathbb{R})$ t.c. $w_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ con $\lambda_1 = 0$, $\lambda_2 = 2$ autovalori

$A \in M_2(\mathbb{R})$ t.c. $w_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ con $\lambda_1 = 0$, $\lambda_2 = 2$ autovalori

$\boxed{\text{modo 1}}$ $A_{\mathcal{E}\mathcal{E}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ impongo $\begin{cases} A w_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ A w_2 = 2 w_2 \end{cases}$

$$\underset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \underset{w_1}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \underset{\lambda_1}{0} \cdot \underset{w_1}{\begin{pmatrix} 1 \\ -3 \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{\text{modo 2}}$ $\mathcal{B} = \{w_1, w_2\} = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix} \right\}$

$$A_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A_{\mathcal{E}\mathcal{E}} = A_{\mathcal{B}\mathcal{E}}(id)\, A_{\mathcal{B}\mathcal{B}}\, A_{\mathcal{E}\mathcal{B}}(id)$$

$$A_{\mathcal{B}\mathcal{E}}(id) = \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$$

$$A_{\mathcal{E}\mathcal{B}}(id) = \frac{1}{-10} \begin{pmatrix} -3 & -1 \\ -1 & 3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 0 & 2 \\ 0 & -6 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 2 & -6 \\ -6 & 18 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$$

F9/E4

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ avt? avl? $D = ?$ $P = ?$

avt = autovettore, avl = autovalore

t.c. $A = P D P^{-1}$

0 è avl

$Av = \lambda v$ se $v \in \ker A \Rightarrow Av = \vec{0} = 0v$

$$Av = \lambda v \quad \text{se } v \in \ker A \Rightarrow Av = \vec{0} = 0 \cdot v \Rightarrow \begin{cases} 0 \text{ è avl} \\ v \text{ è avt} \end{cases}$$

$$\det(x\mathbb{1} - A) = \begin{vmatrix} x-1 & -1 & -1 \\ -1 & x-1 & -1 \\ -1 & -1 & x-1 \end{vmatrix} = (x-1)\begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & x-1 \end{vmatrix} - \begin{vmatrix} -1 & -1 \\ x-1 & -1 \end{vmatrix} =$$

$$= (x-1)\left[(x-1)^2 - 1\right] + \underline{\left[-(x\cancel{-1}) \cancel{-1}\right]} - \underline{\left[\cancel{1} + (x\cancel{-1})\right]}$$

$$= (x-1)(x^2 - 2x) - 2x = x\left[(x-1)(x-2) - 2\right]$$

$$= x\left[x^2 - 3x \cancel{+2} \cancel{-2}\right] = x^2(x-3)$$

$$\Rightarrow \lambda_1 = 0 \quad m.a. = 2$$

$$\lambda_2 = 3 \quad m.a. = 1$$

$$1 \le m.g. \le m.a.$$

$\bullet\ \lambda_1 = 0$

$$A - 0\mathbb{1} = A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \ker(A) = \left\langle \underset{v_1}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \right\rangle$$

$$\Rightarrow \dim(\ker(A)) = 2 \Rightarrow m.g. = 2$$

$\bullet\ \lambda = 3$

$$A - 3\mathbb{1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \qquad \ker(A - 3\mathbb{1}) = \left\langle \underset{v_3}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \right\rangle$$

$$\Rightarrow m.g. = 1$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \mathcal{B} = \{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$P = A_{\mathcal{B}\mathcal{E}}(id) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

F9/E5

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ t.c. $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 \\ 2x_2 \\ x_1 + x_3 \end{pmatrix}$

ⓐ $A_{\mathcal{E}\mathcal{E}}(F) = ?$

$$A_{\mathcal{E}\mathcal{E}}(F) = \left(F\begin{pmatrix}1\\0\\0\end{pmatrix} \; F\begin{pmatrix}0\\1\\0\end{pmatrix} \; F\begin{pmatrix}0\\0\\1\end{pmatrix} \right) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

ⓑ $\ker(F) = ?$

$$\operatorname{Im}(F) = \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \begin{pmatrix}0\\2\\0\end{pmatrix} \cancel{\begin{pmatrix}1\\0\\1\end{pmatrix}} \right\rangle = \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \begin{pmatrix}0\\2\\0\end{pmatrix} \right\rangle \Rightarrow rk(F) = 2 \Rightarrow \dim \ker(F) = 3-2 = 1$$

$$\Rightarrow \ker(F) = \left\langle \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\rangle$$

ⓒ $\exists F^{-1}$?

$$\det(A) = 0 \Rightarrow \nexists F^{-1}$$

ⓓ avl di F?

ⓔ avt? e autospazi?

$$|x\mathbb{1} - A| = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-2 & 0 \\ -1 & 0 & x-1 \end{vmatrix} = (x-2)\left[(x-1)^2 - 1\right] = (x-2)(x^2-2x) = x(x-2)^2$$

$\lambda_1 = 0$, $\quad \lambda_2 = 2$

m.a. $= 1$ $\quad$ m.a. $= 2$

• Se $\lambda_1 = 0$ $\quad A - 0\mathbb{1} = A \quad Ker(A) = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \quad m.g. = 1$

• Se $\lambda_2 = 2$ $\quad A - 2\mathbb{1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad Ker(A - 2\mathbb{1}) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$m.g. = 2$

(F) diagonalizzare F, $D = ?$ $P = ?$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad A = PDP^{-1} \qquad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$P = A_{\mathcal{B}\mathcal{E}}(id) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

F9/E6

$$A_K = \begin{pmatrix} 1 & 2K-4 & -1 \\ K-2 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

(a) K t.c. $v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ è avt. ?

per tali K trovare avt, avl e m.a., m.g.

(b) K t.c. 0 è avl ?

(c) K t.c. A_K è diag. ?

(d) $K = 2$ trovare avt $\in W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ y = 0 \right\}$

(a) $A_K v = \lambda v \iff \underbrace{\begin{pmatrix} 1 & 2K-4 & -1 \\ K-2 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{= \begin{pmatrix} 2 \\ K-2 \\ -2 \end{pmatrix}} = \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ -\lambda \end{pmatrix}$

$\{ 2 = \lambda \qquad \{ \lambda = 2$

$$\Rightarrow \begin{cases} 2 = \lambda \\ K-2 = 0 \\ -2 = -\lambda \end{cases} \Rightarrow \begin{cases} \lambda = 2 \\ K = 2 \end{cases}$$

se $K=2$

$$|x\mathbb{1} - A_2| = \begin{vmatrix} x-1 & 0 & 1 \\ 0 & x & 0 \\ 1 & 0 & x-1 \end{vmatrix} = x\left((x-1)^2-1\right) = x^2(x-2)$$

$\lambda_1 = 0 \quad \lambda_2 = 2$

$m.a. = 2 \quad m.a. = 1$

• se $\lambda_1 = 0$

$$A_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$Ker(A_2) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad m.g. = 2$$

• $\lambda_2 = 2 \Rightarrow m.g. = 1 \qquad Ker(A_2 - 2\mathbb{1}) = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$

b) $\Rightarrow$ imponiamo $\det(A_K) = 0$

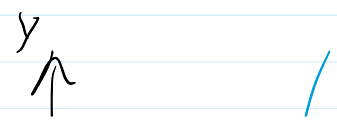
$$\Rightarrow \begin{vmatrix} 1 & 2K-4 & -1 \\ K-2 & 0 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 2(K-2)\begin{vmatrix} K-2 & 0 \\ -1 & 1 \end{vmatrix} = 2(K-2)^2 = 0$$

0 è avl $\Leftrightarrow 2(K-2)^2 = 0 \Leftrightarrow \boxed{K=2}$

c) K t.c. A_K è diag.?

$$P_{A_K}(x) = |x\mathbb{1} - A_K| = \begin{vmatrix} x-1 & 4-2K & 1 \\ 2-K & x & 0 \\ 1 & 0 & x-1 \end{vmatrix} = x\left[(x-1)^2-1\right] - 2(2-K)^2(x-1)$$

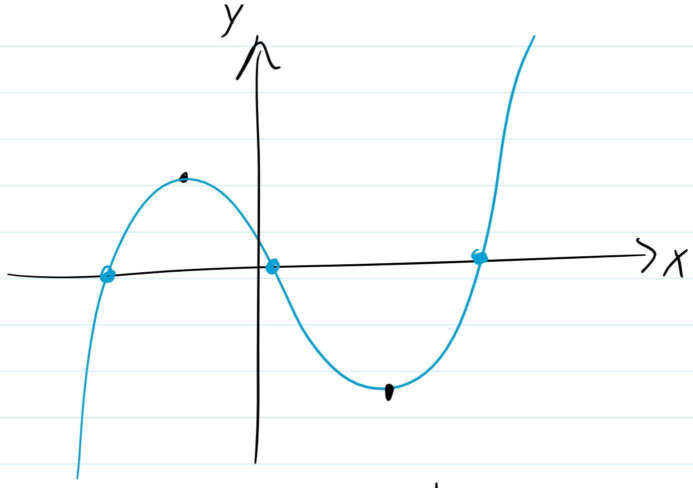
$$= x^2(x-2) - 2(2-K)^2(x-1)$$



$1 \leq m.g. \leq m.a. = 1$

$\Downarrow$

$m.g. = 1$



ci sono sempre 3 soluzioni distinte tranne quando $K=2$

se $K=2 \Rightarrow P_{A_2}(x) = x^2(x-2)$

$\Rightarrow A_K$ è diag. $\forall K$

(d) se $K=2$ $\quad \bullet\ \lambda=0 \quad Ker(A_2) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$\bullet\ \lambda=2 \quad Ker(A_2 - 2\mathbb{1}) = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$

$$\bullet\ \underbrace{\left\{ \begin{pmatrix} a \\ 0 \\ -a \end{pmatrix} \middle| \ a \in \mathbb{R} \right\}}_{\lambda=2} \cup \underbrace{\left\{ \begin{pmatrix} b \\ 0 \\ b \end{pmatrix} \middle| \ b \in \mathbb{R} \right\}}_{\lambda=0}$$

$$\left(Ker(A_2 - 2\mathbb{1}) \cap W \right) \cup \left(Ker(A_2) \cap W \right)$$