2020/05/12
domenica 10 maggio 2020 16:14

# F6/E5

$F_K : \mathbb{R}^3 \to \mathbb{R}^3$ t.c. $A_{\mathcal{E}\mathcal{E}}(F_K) = \begin{pmatrix} K-3 & K & K^2-1 \\ 2 & K-2 & K-1 \\ 0 & 0 & 1 \end{pmatrix} =: A_K$

a) K t.c. $A_K$ iniettiva?

b) K t.c. suriettiva / K t.c. biezione?

c) K t.c. $A_K^{-1}\left(\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}\right) \neq \emptyset$ ?

d) $A_K^{-1}\left(\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}\right) = ?$ per K t.c. infinite sol.

e) $\mathcal{B}$ di $\mathbb{R}^3$ t.c. $A_{\mathcal{B}\mathcal{E}}(F_K) = \begin{pmatrix} 2K-3 & K & K^2-1 \\ K & K-2 & K-1 \\ 0 & 0 & 1 \end{pmatrix}$

a) $\det(A_K) = 0 \Rightarrow rk(A_K) \leq 2 \Rightarrow \dim(Ker(A_K)) = 3 - rk(A_K) \geq 3-2 = 1$

$$0 = \begin{vmatrix} K-3 & K & K^2-1 \\ 2 & K-2 & K-1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} K-3 & K \\ 2 & K-2 \end{vmatrix} = K^2-5K+6-2K = K^2-7K+6 = (K-6)(K-1)$$

$\boxed{K=1,\ K=6}$ → K t.c. $A_K$ NON è iniettiva $\Rightarrow K \in \mathbb{R}\setminus\{1,6\}$

modo 2) risolvere $\left(A_K \,\middle|\, \begin{matrix}0\\0\\0\end{matrix}\right)$

b) è suriettiva $\Leftrightarrow K \in \mathbb{R}\setminus\{1,6\}$

è biezione $\Leftrightarrow K \in \mathbb{R}\setminus\{1,6\}$

c) cerchiamo K t.c. $\begin{pmatrix}1\\-1\\1\end{pmatrix} \in Im(A_K)$

• se $K \in \mathbb{R} \setminus \{1, 6\} \Rightarrow A_K$ suriettiva $\Rightarrow A_K^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} \neq \emptyset$

• se $K=1 \Rightarrow \operatorname{Im}(A_K) = \left\langle \begin{pmatrix}-2\\2\\0\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle = \left\langle \underset{v_1}{\begin{pmatrix}1\\-1\\0\end{pmatrix}}\underset{v_2}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \right\rangle \ni \begin{pmatrix}1\\-1\\1\end{pmatrix} = v_1 + v_2$

• $K=6 \Rightarrow \operatorname{Im}(A_K) = \left\langle \begin{pmatrix}3\\2\\0\end{pmatrix}\begin{pmatrix}6\\4\\0\end{pmatrix}\begin{pmatrix}35\\5\\1\end{pmatrix} \right\rangle = \left\langle \underset{v_1}{\begin{pmatrix}3\\2\\0\end{pmatrix}}\underset{v_2}{\begin{pmatrix}35\\5\\1\end{pmatrix}} \right\rangle \not\ni \begin{pmatrix}1\\-1\\1\end{pmatrix}$

$$\lambda_1 v_1 + \lambda_2 v_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix} \Leftrightarrow \left(\begin{array}{cc|c} 3 & 35 & 1 \\ 2 & 5 & -1 \\ 0 & 1 & 1 \end{array}\right) \quad \begin{cases} 3\lambda_1 + 35\lambda_2 = 1 \\ 2\lambda_1 + 5\lambda_2 = -1 \\ \lambda_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} -9 + 35 = 1 \\ \lambda_1 = -3 \\ \lambda_2 = 1 \end{cases} \quad ↯$$

d) $K=1$

• se $K=6 \Rightarrow A_K^{-1}(\vdots) = \emptyset$ ↯

• se $K \neq 1, K \neq 6 \Rightarrow \operatorname{Ker}(A_K) = \langle 0 \rangle$

$A_K^{-1}\begin{pmatrix}1\\-1\\1\end{pmatrix} \Rightarrow$ vogliamo risolvere $A_K v = \begin{pmatrix}1\\-1\\1\end{pmatrix}$

$\Updownarrow$

$\left(A_K \middle| \begin{matrix}1\\-1\\1\end{matrix}\right)$

$rk(A_K) = 3 \Rightarrow$ per R-C $\exists!$ soluzione ↯

$\Rightarrow$ rimane solo $K=1$

(e) $\mathcal{B}$ di $\mathbb{R}^3$ t.c. $A_{\mathcal{B}\mathcal{E}}(F_K) = \begin{pmatrix} 2K-3 & K & K^2-1 \\ K & K-2 & K-1 \\ 0 & 0 & 1 \end{pmatrix}$ ?

$$A_{\mathcal{B}\mathcal{E}}(F_K) = A_{\mathcal{E}\mathcal{E}}(F_K) \cdot A_{\mathcal{B}\mathcal{E}}(id)$$

$$\begin{pmatrix} 2K-3 & K & K^2-1 \\ K & K-2 & K-1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} K-3 & K & K^2-1 \\ 2 & K-2 & K-1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

## F8/E3

$$A_K = \begin{pmatrix} 1 & 2 & K \\ K-1 & 1 & 2 \\ 1 & -1 & 2-K \end{pmatrix}$$

(a) K t.c. NO iniett.

(b) per tali K base del Ker, Im

(c) per K t.c. è iniett. calcolare $A_K^{-1}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

(a) $\det(A_K) = 0 \iff$

$$0 = \begin{vmatrix} 1 & 2 & K \\ K-1 & 1 & 2 \\ 1 & -1 & 2-K \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 2-K \end{vmatrix} - (K-1)\begin{vmatrix} 2 & K \\ -1 & 2-K \end{vmatrix} + \begin{vmatrix} 2 & K \\ 1 & 2 \end{vmatrix}$$

$$= \underline{2-k+2} - (k-1)(4-2k+k) + \underline{4-k}$$

$$= 2(4-k) - (k-1)(4-k)$$

$$= (4-k)(2-k+1) = (4-k)(3-k)$$

$\Rightarrow$ $k=4$, $k=3$

(b) • $k=4$ $\Rightarrow$

$$\begin{array}{c} \\ A_4 = \end{array} \begin{array}{ccc} & v_2 & v_3 = 2v_2 \\ \end{array}$$

$$A_4 = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix} \Rightarrow Im(A_4) = \left\langle \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \boxed{\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}} \right\rangle$$

$v_1$ $v_2$ $v_3$ $= 2v_2$

$$= \left\langle \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$\Downarrow$

$rk(A_4) = 2$

$\Rightarrow \dim(\ker(A_4)) = 3-2 = 1$

$$Ker(A_4) = \left\langle \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\rangle$$

$$\left(A_4 \middle| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right)$$

$$A_4 x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

• $k=3$ $\Rightarrow$

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix} \Rightarrow Im(A_3) = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \cancel{\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}} \right\rangle$$

$u_1$ $u_2$ $u_3$

$$= \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$$Ker(A_3) = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & -1 \end{array}\right) \to \begin{array}{c} III \\ II-2III \\ I-III \end{array} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{array}\right) \Rightarrow \begin{cases} x-y=-1 \\ 3y=4 \end{cases}$$

$$\Rightarrow \begin{cases} x=-1+\frac{4}{3}=\frac{1}{3} \\ y=\frac{4}{3} \end{cases}$$

$$\Rightarrow \frac{1}{3}u_1+\frac{4}{3}u_2=u_3$$

$$\Rightarrow \boxed{u_1+4u_2-3u_3=0}$$

(c) per $K \in \mathbb{R}\setminus\{3,4\}$ trovare $\boxed{A_K^{-1}\begin{pmatrix}1\\-2\\0\end{pmatrix}}$

$\Updownarrow$

trovare $\begin{pmatrix}x\\y\\z\end{pmatrix}$ t.c. $A_K\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}$

$\Rightarrow$ cerchiamo $\begin{pmatrix}x\\y\\z\end{pmatrix}=A_K^{-1}\begin{pmatrix}1\\-2\\0\end{pmatrix}$

$$A_K^{-1}=\frac{1}{\det(A_K)}S^T \qquad \det(A_K)=(K-4)(K-3)$$

$$A_K=\begin{pmatrix}1 & 2 & K \\ K-1 & 1 & 2 \\ 1 & -1 & 2-K\end{pmatrix} \qquad S=\begin{pmatrix}-K+4 & K^2-3K+4 & -K \\ K-4 & -2K+2 & 3 \\ -K+4 & K^2-K-2 & -2K+3\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{(k-3)(k-4)} \begin{pmatrix} -k+4 & k-4 & -k+4 \\ k^2-3k+4 & -2k+2 & k^2-k-2 \\ -k & 3 & -2k+3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{(k-3)(k-4)} \begin{pmatrix} -3(k-4) \\ k(k-1) \\ -(k+6) \end{pmatrix}$$

## F8/E5

$$U = \left\langle \underset{u_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}} \right\rangle \qquad V = \left\langle \underset{v_3}{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}} \right\rangle$$

a) $U \oplus V = \mathbb{R}^3$ ?

b) calcolare per $\varepsilon$ $\sigma: \mathbb{R}^3 \to \mathbb{R}^3$ simmetria con asse $U$ parall. $V$

a) $U: \{ x - 2y + z = 0$ $\qquad v_3 \notin U \qquad 0 - 2(1) + (-2) = -4 \neq 0$

modo 1

$$\Downarrow$$

$$v_3 \notin U$$

modo 2

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 2 - 2(1-2) = 2 + 2 = 4 \neq 0$$

✓

b) modo 1

$$\mathcal{B} = \Big\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}}_{V} \Big\}$$

$$\boxed{\begin{aligned} \sigma : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ u+v &\longmapsto u-v \end{aligned}}$$

$$A_{\mathcal{B}\mathcal{B}}(\sigma) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\sigma}$$

$$U \begin{cases} \sigma(u_1) = u_1 \\ \sigma(u_2) = u_2 \end{cases}$$

$$V \ \{ \ \sigma(v_3) = -v_3$$

$$A_{\mathcal{E}\mathcal{E}}(\sigma) = A_{\mathcal{B}\mathcal{E}}(id)\, A_{\mathcal{B}\mathcal{B}}(\sigma)\, A_{\mathcal{E}\mathcal{B}}(id)$$

$$\begin{tikzcd} \mathbb{R}^3_{\mathcal{E}} \arrow[r, "A_{\mathcal{E}\mathcal{E}}(\sigma)"] \arrow[d, "A_{\mathcal{E}\mathcal{B}}(id)"'] & \mathbb{R}^3_{\mathcal{E}} \\ \mathbb{R}^3_{\mathcal{B}} \arrow[r, "A_{\mathcal{B}\mathcal{B}}(\sigma)"] & \mathbb{R}^3_{\mathcal{B}} \arrow[u, "A_{\mathcal{B}\mathcal{E}}(id)"'] \end{tikzcd}$$

$$A_{\mathcal{B}\mathcal{E}}(id) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} =: A$$

$$\det(A) = 2 - 2(1-2) = 4$$

$$A_{\mathcal{E}\mathcal{B}}(id) = \left(A_{\mathcal{B}\mathcal{E}}(id)\right)^{-1} = \frac{1}{4} \begin{pmatrix} -2 & 4 & 2 \\ 3 & -2 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

$$\Rightarrow A_{\mathcal{E}\mathcal{E}}(\sigma) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \left( \frac{1}{4} \begin{pmatrix} -2 & 4 & 2 \\ 3 & -2 & -1 \\ -1 & 2 & -1 \end{pmatrix} \right)$$

$$\Rightarrow M^{\mathcal{E}}_{\mathcal{E}}(\sigma) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\left(\frac{1}{4}\begin{pmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \end{pmatrix}\right)$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -2 & 4 & 2 \\ 3 & -2 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ -1 & 2 & 0 \end{pmatrix}$$

$\boxed{\text{modo 2:}}$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} =: A \quad \text{t.c.}$$

$$\left.\begin{aligned} A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ A\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \end{aligned}\right| \in U$$

$$A\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \Bigg| \in V$$

$\boxed{\text{F8/E6}} \longrightarrow$ esercizio per casa