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FS/E3

$\mathbb{R}^4$ $\quad U: \begin{cases} x+y+z+w=0 \\ 2x+3y-z+w=0 \end{cases}$ $\quad V\left\langle \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\rangle$

ⓐ $k=?$ t.c. $\phi: \mathbb{R}^4 \to \mathbb{R}^2$ Lineare t.c.

$\ker\phi = U$, $\phi\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$, $\phi\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\k\end{pmatrix}$, $\phi\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = \begin{pmatrix}-2\\-k\end{pmatrix}$

ⓑ $A_{\mathcal{E}_4 \mathcal{E}_2}(\phi) = ?$

ⓒ $rk A = ?$

ⓓ $\phi^{-1}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}\right) = ?$

ⓐ ϕ lineare $\Rightarrow$ $\phi\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix} a_1x+b_1y+c_1z+d_1w \\ a_2x+b_2y+c_2z+d_2w \end{pmatrix}$

$$= \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix} \begin{pmatrix}x\\y\\z\\w\end{pmatrix}$$

- $\ker\phi = U$

$$U = \left\{ \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{cases} x+y+z+w=0 \\ 2x+3y-z+w=0 \end{cases} \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \underset{II-2I}{} \begin{cases} x=-y-z-w \\ y-3z-w=0 \end{cases} \right\}$$

$\hookrightarrow \begin{cases} x=-(3z+w)-z-w=-4z-2w \\ y=3z+w \end{cases}$

$$= \left\{ \begin{pmatrix}-4z-2w\\3z+w\\z\\w\end{pmatrix} \;\middle|\; z,w \in \mathbb{R} \right\}$$

$$= \left\langle \begin{pmatrix}-4\\3\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\1\\0\\1\end{pmatrix} \right\rangle$$

$u_1 \quad u_2$ t.c. $\phi(u_1) = \phi(u_2) = 0$

$$u_1 \qquad u_2 \qquad \text{t.c. } \phi(u_1)=\phi(u_2)=0$$

$$v \in U \Rightarrow v = \lambda_1 u_1 + \lambda_2 u_2$$

$$\Rightarrow \phi(v) = \lambda_1 \phi(u_1) + \lambda_2 \phi(u_2) = 0+0 = 0$$

$$\phi\begin{pmatrix}-4\\3\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{cases} -4a_1+3b_1+c_1=0 \\ -4a_2+3b_2+c_2=0 \end{cases}$$

$$\phi\begin{pmatrix}-2\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{cases} -2a_1+b_1+d_1=0 \\ -2a_2+b_2+d_2=0 \end{cases}$$

$$\left(\begin{array}{cccc|c} -4 & 3 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\mathrm{I}-2\mathrm{II}]{\mathrm{II}} \left(\begin{array}{cccc|c} -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \end{array}\right) \qquad i=1,2$$

$$\begin{cases} -2a_i+b_i+d_i=0 \\ b_i+c_i-2d_i=0 \end{cases}$$

$$\Rightarrow \begin{cases} b_i = 2a_i - d_i \\ c_i = 2d_i - b_i \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} b_i = 2a_i - d_i \\ c_i = 2d_i - (2a_i - d_i) = 3d_i - 2a_i \end{cases}} \qquad i=1,2$$

$$\Rightarrow \phi\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix} a_1 & 2a_1-d_1 & 3d_1-2a_1 & d_1 \\ a_2 & 2a_2-d_2 & 3d_2-2a_2 & d_2 \end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$$

- $$\boxed{\phi\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\k\end{pmatrix}} \Rightarrow \begin{pmatrix} a_1 & 2a_1-d_1 & 3d_1-2a_1 & d_1 \\ a_2 & 2a_2-d_2 & 3d_2-2a_2 & d_2 \end{pmatrix}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\k\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3d_1-2a_1+d_1 \\ 3d_2-2a_2+d_2 \end{pmatrix} = \begin{pmatrix}1\\k\end{pmatrix}$$

$$\begin{cases} 4d_1 - 2a_1 = 1 \end{cases} \qquad \begin{cases} a_1 = 4d_1 - 1 \end{cases}$$

$$\Rightarrow \begin{cases} 4d_1 - 2a_1 = 1 \\ 4d_2 - 2a_2 = K \end{cases} \Rightarrow \begin{cases} a_1 = \frac{4d_1 - 1}{2} \\ a_2 = \frac{4d_2 - K}{2} \end{cases}$$

$$\phi\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2d_1 - \frac{1}{2} & 3d_1 - 1 & 1 - d_1 & d_1 \\ 2d_2 - \frac{K}{2} & 3d_2 - K & K - d_2 & d_2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- $\phi\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -K \end{pmatrix}$

$$\begin{cases} 2d_1 - \frac{1}{2} - (1 - d_1) = -2 \\ 2d_2 - \frac{K}{2} - (\cancel{K} - d_2) = \cancel{-}K \end{cases}$$

$$\Rightarrow \begin{cases} 3d_1 = -\frac{1}{2} \\ 3d_2 = \frac{K}{2} \end{cases} \Rightarrow \begin{cases} d_1 = -\frac{1}{6} \\ d_2 = \frac{K}{6} \end{cases}$$

$$\Rightarrow \phi\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} & -\frac{3}{2} & \frac{7}{6} & -\frac{1}{6} \\ -\frac{K}{6} & -\frac{K}{2} & \frac{5}{6}K & \frac{K}{6} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

- $\phi\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} -\frac{5}{6} - \left(-\frac{3}{2}\right) + \frac{7}{6} - \left(-\frac{1}{6}\right) = 2 \\ -\frac{K}{6} - \left(-\frac{K}{2}\right) + \frac{5}{6}K - \left(\frac{K}{6}\right) = 1 \end{cases}$

$$\Rightarrow \begin{cases} \frac{\overbrace{-5+9+7+1}^{12}}{6} = 2 \quad \checkmark \\ K\left(\frac{-1+3+5-1}{6}\right) = 1 \end{cases} \Rightarrow K = 1$$

$$\phi\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}-\frac{5}{6} & -\frac{3}{2} & \frac{7}{6} & -\frac{1}{6}\\ -\frac{1}{6} & -\frac{1}{2} & \frac{5}{6} & \frac{1}{6}\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$$

b) $$\underbrace{A_{\mathcal{E}_4\mathcal{E}_2}(\phi)}_{\overset{\shortparallel}{A}}=\begin{pmatrix}-\frac{5}{6} & -\frac{3}{2} & \frac{7}{6} & -\frac{1}{6}\\ -\frac{1}{6} & -\frac{1}{2} & \frac{5}{6} & \frac{1}{6}\end{pmatrix}$$

c) $\operatorname{rk} A = ?$ $\qquad \operatorname{Im}(\phi)\subseteq\mathbb{R}^2 \qquad \phi:\mathbb{R}^4\to\mathbb{R}^2$

$$\mathbb{R}^2\supseteq\operatorname{Im}(\phi)\supseteq\Big\langle\underset{v_1}{\begin{pmatrix}2\\1\end{pmatrix}}\underset{v_2}{\begin{pmatrix}1\\1\end{pmatrix}}\Big\rangle=\Big\langle\underset{v_1-v_2}{\begin{pmatrix}1\\0\end{pmatrix}}\begin{pmatrix}1\\1\end{pmatrix}\Big\rangle=\Big\langle\begin{pmatrix}1\\0\end{pmatrix}\underset{2v_2-v_1}{\begin{pmatrix}0\\1\end{pmatrix}}\Big\rangle=\mathbb{R}^2$$

$\Rightarrow \operatorname{Im}\phi=\mathbb{R}^2 \quad\Rightarrow\quad \operatorname{rk}(A)=\dim(\operatorname{Im}\phi)=\dim(\mathbb{R}^2)=2$

d) $\phi^{-1}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}\right) \qquad \boxed{\phi\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}} \iff A\cdot v=\begin{pmatrix}1\\2\end{pmatrix}$

$\phi\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=A\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \qquad \operatorname{Ker} A = U$

$$\phi\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix},\quad \phi\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\Rightarrow \begin{pmatrix}1\\2\end{pmatrix}=3\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}2\\1\end{pmatrix}=$$

$$=3\phi\begin{pmatrix}0\\0\\1\\1\end{pmatrix}-\phi\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$=\phi\left(\begin{pmatrix}0\\0\\3\\3\end{pmatrix}-\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right)=\phi\begin{pmatrix}-1\\1\\2\\4\end{pmatrix}$$

$$\Rightarrow \phi^{-1}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}\right)=\begin{pmatrix}-1\\1\\2\\4\end{pmatrix}+\left\langle\begin{pmatrix}-4\\3\\1\\0\end{pmatrix}\begin{pmatrix}-2\\1\\0\\1\end{pmatrix}\right\rangle$$

F6/E3

ⓐ $A_{\mathcal{E}\mathcal{E}}(F_a) = ?$

ⓑ $a = ?$ t.c. F_a NON è iniettiva

$\ker F_a = ?$ $\exists$ altri a t.c. $\ker(F_a) \neq \langle 0 \rangle$?

$a \in \mathbb{R}$, $F_a\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ x+y+az \\ z \end{pmatrix}$

$F: \mathbb{R}^3 \to \mathbb{R}^3$

ⓐ $F_a\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ x+y+az \\ z \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 1 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Rightarrow A_{\mathcal{E}\mathcal{E}}(F_a) = \begin{pmatrix} a & 0 & 0 \\ 1 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$

ⓑ F_a NON è iniettiva $\Longleftrightarrow \ker(F_a) \neq \langle 0 \rangle$

$\begin{pmatrix} a & 0 & 0 \\ 1 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow \underbrace{\left(\begin{array}{ccc|c} a & 0 & 0 & 0 \\ 1 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)}_{A \quad b}$

se $a \neq 0 \Rightarrow$ $\begin{array}{c} \\ II - \frac{I}{a} \\ \\ \end{array} \left(\begin{array}{ccc|c} a & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow \ker(F_a) = \langle 0 \rangle$

se $a = 0 \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \ker(F_0) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{cases} x+y=0 \\ z=0 \end{cases} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{cases} x=-y \\ z=0 \end{cases} \right\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$

FS/E6

$\phi: \mathbb{R}^{\leq 4}[x] \longrightarrow \mathbb{R}^{\leq 4}[x]$ t.c. $\phi(P(x)) = \frac{d}{dx}(P(x))$

a) ϕ è lineare?

b) $\operatorname{Ker}\phi = ?$ $\operatorname{Im}\phi = ?$

c) $A_{\mathcal{E}\mathcal{E}}(\phi) = ?$ $\mathcal{E} = \{1, x, x^2, x^3, x^4\}$

a) $v_1 = a_1 + b_1 x + c_1 x^2 + d_1 x^3 + e_1 x^4$

$v_2 = a_2 + b_2 x + c_2 x^2 + d_2 x^3 + e_2 x^4$

$\lambda \in \mathbb{R}$

$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$, $\phi(\lambda v_1) = \lambda \phi(v_1)$

$$\phi(v_1+v_2) = \phi\left((a_1+a_2) + (b_1+b_2)x + (c_1+c_2)x^2 + (d_1+d_2)x^3 + (e_1+e_2)x^4\right) =$$

$$= \frac{d}{dx}\left((a_1+a_2) + (b_1+b_2)x + (c_1+c_2)x^2 + (d_1+d_2)x^3 + (e_1+e_2)x^4\right)$$

$$= (b_1+b_2) + 2(c_1+c_2)x + 3(d_1+d_2)x^2 + 4(e_1+e_2)x^3$$

$$= \frac{d}{dx}\left(a_1 + b_1 x + c_1 x^2 + d_1 x^3 + e_1 x^4\right) + \frac{d}{dx}\left(a_2 + b_2 x + c_2 x^2 + d_2 x^3 + e_2 x^4\right)$$

$$= \phi(v_1) + \phi(v_2)$$

$$\phi(\lambda v_1) = \frac{d}{dx}\left(\lambda\left(a_1 + b_1 x + c_1 x^2 + d_1 x^3 + e_1 x^4\right)\right)$$

$$= \lambda \frac{d}{dx}\left(a_1 + b_1 x + c_1 x^2 + d_1 x^3 + e_1 x^4\right) = \lambda \phi(v_1)$$

SI, ϕ è lineare

(b) $\operatorname{Ker}\phi, \operatorname{Im}\phi = ?$ $\qquad P(x) = a + bx + cx^2 + dx^3 + ex^4$

$$\operatorname{Ker}\phi = \left\{ P(x) \in \mathbb{R}^{\leq 4}[x] \;\middle|\; \frac{d}{dx}(P(x)) = 0 \right\}$$

$\mathbb{R}^{\leq 4}[x]$

$\Downarrow$

$$0 = \frac{d}{dx}\left(a + bx + cx^2 + dx^3 + ex^4\right) = b + 2cx + 3dx^2 + 4ex^3$$

$$\Rightarrow \begin{cases} b = 0 \\ 2c = 0 \\ 3d = 0 \\ 4e = 0 \end{cases} \Rightarrow$$

$$= \left\{ a + bx + cx^2 + dx^3 + ex^4 \in \mathbb{R}^{\leq 4}[x] \;\middle|\; b = c = d = e = 0 \right\}$$

$$= \{ a \mid a \in \mathbb{R} \} = \langle 1 \rangle = \operatorname{Ker}\phi \qquad \dim(\operatorname{Ker}\phi) = 1$$

$\operatorname{Im}\phi = ?$ $\qquad \operatorname{Im}\phi = \{ \phi(v) \mid v \in \mathbb{R}^{\leq 4}[x] \}$

$$= \left\{ b + 2cx + 3dx^2 + 4ex^3 \in \mathbb{R}^{\leq 4}[x] \;\middle|\; a + bx + cx^2 + dx^3 + ex^4 \in \mathbb{R}^{\leq 4}[x] \right\}$$

$$= \left\{ b + 2cx + 3dx^2 + 4ex^3 \in \mathbb{R}^{\leq 4}[x] \;\middle|\; a, b, c, d, e \in \mathbb{R} \right\}$$

$$= \left\{ b + c(2x) + d(3x^2) + e(4x^3) \in \mathbb{R}^{\leq 4}[x] \;\middle|\; a, b, c, d, e \in \mathbb{R} \right\}$$

$$= \langle 1, 2x, 3x^2, 4x^3 \rangle = \langle 1, x, x^2, x^3 \rangle = \operatorname{Im}\phi \Rightarrow \dim(\operatorname{Im}\phi) = 4$$

(c) $\underbrace{A_{\mathcal{E}\mathcal{E}}(\phi)}_{A} = ?$ $\qquad \mathcal{E} = \{1, x, x^2, x^3, x^4\}$

$$A = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \end{pmatrix} \qquad A\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = u_1 \qquad A\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} = u_2 \ \ldots$$

$$u_1 = \phi\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}$$

$$\phi(1) = \frac{d}{dx}(1) = 0$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$u_2 = \phi\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$$

$$\phi(x) = \frac{d}{dx}(x) = 1$$

$$u_3 = \phi\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\\0\\0\end{pmatrix} \qquad u_4 = \phi\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\3\\0\\0\end{pmatrix} \qquad u_5 = \phi\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\4\\0\end{pmatrix}$$

$$\phi(x^2) = \frac{d}{dx}(x^2) = 2x \qquad \phi(x^3) = 3x^2 \qquad \frac{d}{dx}(x^4) = 4x^3$$

$\boxed{\text{es casa}}$

trovare $\phi : \mathbb{R}^{\leq 5}[x] \to \mathbb{R}^{\leq 5}[x]$ t.c. $\phi(p(x)) = \frac{d^2}{dx^2}(p(x))$

F6/E1

$$\begin{cases} x + y + z = 2k \\ (k-1)x + y - z = 1 \\ x + ky + z = k+1 \end{cases}$$

(a) sol. per $k=0$

(b) $k = ?$ t.c. $\exists$ più di una sol.

(c) $k = ?$ t.c. $\exists!$ la soluzione sol: ⓐ $K=0 \Rightarrow \nexists$ sol.

ⓑ $K=1 \Rightarrow \exists$ più di una sol.

ⓒ $\exists!$ sol. $\Leftrightarrow K \neq 0$ e $K \neq 1$