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F5/E1

$$\phi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \qquad \phi\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+2z \\ y-z+w \\ 2x+y-w \end{pmatrix}$$

a) ϕ è lin.?

b) $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$?

c) $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad A_{\mathcal{B}\mathcal{E}}(\phi)$

d) $W \leq \mathbb{R}^4$ t.c. $\dim(\phi(W)) = \dim W$, esempio con dim. max

a) • $\phi(v+w) = \phi(v) + \phi(w)$

• $\phi(\alpha v) = \alpha\phi(v)$

$$\underbrace{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}}_{v}, \underbrace{\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}}_{u} \in \mathbb{R}^4, \alpha \in \mathbb{R}$$

• $$\phi(v+u) = \phi\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \\ w_1+w_2 \end{pmatrix} = \begin{pmatrix} (x_1+x_2)+(y_1+y_2)+2(z_1+z_2) \\ (y_1+y_2)-(z_1+z_2)+(w_1+w_2) \\ 2(x_1+x_2)+(y_1+y_2)-(w_1+w_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1+y_1+2z_1 \\ y_1-z_1+w_1 \\ 2x_1+y_1-w_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2+2z_2 \\ y_2-z_2+w_2 \\ 2x_2+y_2-w_2 \end{pmatrix}$$

$$= \phi\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix} + \phi\begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} = \phi(v) + \phi(u)$$

• $$\phi(\alpha v) = \phi\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \\ \dots \end{pmatrix} = \begin{pmatrix} \alpha x + \alpha y + 2\alpha z \\ \alpha y - \alpha z + \alpha w \\ \dots \end{pmatrix} = \alpha\begin{pmatrix} x+y+2z \\ y-z+w \\ 2x+y-w \end{pmatrix}$$

$$\bullet\ \phi(\alpha v) = \phi\begin{pmatrix} \alpha y \\ \alpha t \\ \alpha w \end{pmatrix} = \begin{pmatrix} \dots \\ \alpha y - \alpha z + \alpha w \\ 2\alpha x + \alpha y - \alpha w \end{pmatrix} = \alpha \begin{pmatrix} \dots \\ y - z + w \\ 2x + y - w \end{pmatrix} = \alpha \phi(v) \quad \checkmark$$

(b) $$\phi(v) = \begin{pmatrix} x + y + 2z \\ y - z + w \\ 2x + y - w \end{pmatrix}$$

$$\mathrm{Ker}\,\phi = \{ v \in \mathbb{R}^4 \mid \phi(v) = 0 \in \mathbb{R}^3 \}$$

$$\phi\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x + y + 2z \\ y - z + w \\ 2x + y - w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x + y + 2z = 0 \\ y - z + w = 0 \\ 2x + y - w = 0 \end{cases} \Rightarrow \left(\underbrace{\begin{matrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 0 & -1 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}}_{b} \right)$$

$$\xrightarrow[III - 2I]{} \left(\begin{matrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & -4 & -1 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right) \xrightarrow[III + II]{} \left(\begin{matrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -5 & 0 \end{matrix} \;\middle|\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right)$$

$$\begin{cases} x + y + 2z = 0 \\ y - z + w = 0 \\ -5z = 0 \end{cases} \Rightarrow \begin{cases} x = -y = w \\ y = -w \\ z = 0 \end{cases}$$

$$\Rightarrow \mathrm{Ker}\,\phi = \left\{ \begin{pmatrix} w \\ -w \\ 0 \\ w \end{pmatrix} \;\middle|\; w \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\mathrm{Im}\,\phi = \left\{ \phi(v) \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \right\}$$

$$= \left\{ \begin{pmatrix} x+y+2z \\ y-z+w \\ 2x+y-w \end{pmatrix} \middle| \; x,y,z,w \in \mathbb{R} \right\}$$

$$= \left\{ x\begin{pmatrix}1\\0\\2\end{pmatrix} + y\begin{pmatrix}1\\1\\1\end{pmatrix} + z\begin{pmatrix}2\\-1\\0\end{pmatrix} + w\begin{pmatrix}0\\1\\-1\end{pmatrix} \middle| \; x,y,z,w\in\mathbb{R}\right\}$$

$$= \left\langle \underset{v_1}{\begin{pmatrix}1\\0\\2\end{pmatrix}} \underset{v_2}{\cancel{\begin{pmatrix}1\\1\\1\end{pmatrix}}} \underset{v_3}{\begin{pmatrix}2\\-1\\0\end{pmatrix}} \underset{v_4}{\begin{pmatrix}0\\1\\-1\end{pmatrix}} \right\rangle \qquad v_1 + v_4 = \begin{pmatrix}1\\1\\1\end{pmatrix} = v_2$$

$$= \left\langle \begin{pmatrix}1\\0\\2\end{pmatrix} \begin{pmatrix}2\\-1\\0\end{pmatrix} \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\rangle$$

$$\dim(\mathbb{R}^4) = \dim(\mathrm{Ker}\,\phi) + \dim(\mathrm{Im}\,\phi)$$

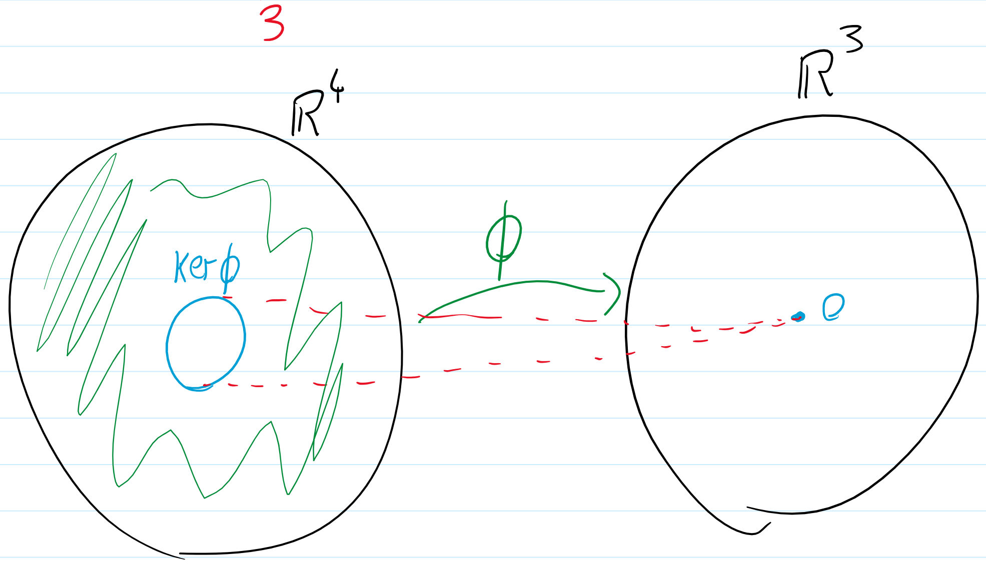
$$4 \quad = \quad 1$$

$$\Rightarrow \dim(\mathrm{Im}\,\phi) = 4 - 1 = 3$$

d) $W \subseteq \mathbb{R}^4$ t.c. $\dim(\phi(W)) = \dim W$ e $\dim W = \max$

$$\dim(\phi(W)) \leq \dim(\mathrm{Im}\,\phi) = 3 \qquad \phi(W) \subseteq \mathrm{Im}\,\phi$$

$\dim W = 3$



Sia W t.c. $W \oplus \operatorname{Ker}\phi = \mathbb{R}^4$

$$\dim W + \dim \operatorname{Ker}\phi = \dim(\mathbb{R}^4)$$
$$\phantom{\dim W + \dim \operatorname{Ker}\phi} 1 \quad = \quad 4$$

$\Rightarrow \dim W = 3$

$$\operatorname{Ker}\phi = \left\langle \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{v} \right\rangle \qquad W = \left\langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{e_1} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{e_3} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{e_4} \right\rangle$$

$$\lambda_1 v + \lambda_2 e_1 + \lambda_3 e_3 + \lambda_4 e_4 = 0$$

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ -\lambda_1 = 0 \\ \lambda_3 = 0 \\ \lambda_1 + \lambda_4 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 \dots = \lambda_4 = 0$$

ora verifichiamo che $\dim W = \dim(\phi(W))$

$$v \in W \Rightarrow v = \lambda_1 e_1 + \lambda_2 e_3 + \lambda_3 e_4$$

$$\Rightarrow \phi(v) = \lambda_1 \phi(e_1) + \lambda_2 \phi(e_3) + \lambda_3 \phi(e_4)$$
$$\cap$$
$$\phi(W) = \langle \phi(e_1), \phi(e_3), \phi(e_4) \rangle$$
$$\| $$
$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle = \operatorname{Im}\phi$$

$$\Rightarrow \dim(\phi(W)) = \dim(\operatorname{Im}\phi) = 3 = \dim W$$

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(c) $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\quad \mathcal{E} = \left\{ \begin{array}{ccc} w_1 & w_2 & w_3 \\ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} & \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array} \right\}$ $\quad A_{\mathcal{B}\mathcal{E}}(\phi) = ?$

c) $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\quad \mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\quad A_{\mathcal{B}\mathcal{E}}(\phi) = ?$

$v_1 \quad v_2 \quad v_3 \quad v_4$

$A_{\mathcal{B}\mathcal{E}}(\phi) \in M_{3,4}(\mathbb{R})$

$$\phi\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\hookrightarrow$ in $\mathcal{E}$ ha coe. $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

coe. in $\mathcal{B}$

$$\underbrace{A_{\mathcal{B}\mathcal{E}}(\phi) = \begin{pmatrix} a & \cdot & \cdot & \cdot \\ b & \cdot & \cdot & \cdot \\ c & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\phi(v_1)} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

coe. in $\mathcal{E}$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1\, v_1 + 0 \cdot v_2 + 0 \cdot v_3 + 0 \cdot v_4$$

nella base $\mathcal{B}$ le coordinate di v_1 sono $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0 \cdot v_1 + 1 \cdot v_2 + 0 v_3 + 0 \cdot v_4$$

le coe. di v_2 in $\mathcal{B}$ sono $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$$\underbrace{A_{\mathcal{B}\mathcal{E}}(\phi)}_{!!} = \begin{pmatrix} 1 & 1 & 2/3 & 2/3 \\ -1 & -2 & -4/3 & -1/3 \\ 3 & 2 & -1/3 & 2/3 \end{pmatrix}$$

A

$$\mathcal{B} = \left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \right\} \quad \mathcal{E} = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}-1\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

$$\phi(v_2) = \begin{pmatrix}3\\0\\1\end{pmatrix} = 1\begin{pmatrix}1\\2\\1\end{pmatrix} - 2\begin{pmatrix}-1\\1\\1\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\phi(v_3) = \begin{pmatrix}2\\0\\-1\end{pmatrix} = \frac{2}{3}\left(\begin{pmatrix}1\\2\\1\end{pmatrix} - 2\begin{pmatrix}-1\\1\\1\end{pmatrix}\right) - \frac{1}{3}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \frac{2}{3}\begin{pmatrix}1\\2\\1\end{pmatrix} - \frac{4}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\phi(v_4) = \begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{2}{3}\begin{pmatrix}1\\2\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

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es/extra

a) $F: \mathbb{R}^3 \to \mathbb{R}^3$ lineare t.c. $\operatorname{Ker} F = \left\langle \begin{pmatrix}2\\0\\-1\end{pmatrix} \right\rangle$ $\operatorname{Im} F = \left\langle \begin{pmatrix}2\\0\\-1\end{pmatrix}\begin{pmatrix}2\\1\\2\end{pmatrix} \right\rangle$

F è unica? perché?

b) trovare $A_{\mathcal{E}\mathcal{E}}(F)$?

c) $F^{-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$, $F^{-1}\begin{pmatrix}0\\-1\\-3\end{pmatrix}$?

a) F è lineare
$$F\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a_1x + b_1y + c_1z\\ a_2x + b_2y + c_2z\\ a_3x + b_3y + c_3z\end{pmatrix}$$

imposto linearità

$$\operatorname{Ker} F = \left\langle \begin{pmatrix}2\\0\\-1\end{pmatrix} \right\rangle$$

$$= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, F\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \right\} = \left\{ \lambda\begin{pmatrix}2\\0\\-1\end{pmatrix} \,\middle|\, F\left(\lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}\right) = \begin{pmatrix}0\\0\\0\end{pmatrix} \ \forall \lambda \in \mathbb{R} \right\}$$

$$\lambda f\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow f\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}2a_1-c_1\\2a_2-c_2\\2a_3-c_3\end{pmatrix} \Rightarrow \begin{cases}c_1=2a_1\\c_2=2a_2\\c_3=2a_3\end{cases} \Rightarrow \boxed{f\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1x+b_1y+2a_1z\\a_2x+b_2y+2a_2z\\a_3x+b_3y+2a_3z\end{pmatrix}}$$

$$\operatorname{Im} f=\left\langle\begin{pmatrix}2\\0\\-1\end{pmatrix}\begin{pmatrix}2\\1\\2\end{pmatrix}\right\rangle$$

$$=\{f(v)\mid v\in\mathbb{R}^3\}=\left\{\begin{pmatrix}a_1x+b_1y+2a_1z\\a_2x+b_2y+2a_2z\\a_3x+b_3y+2a_3z\end{pmatrix}\middle|\ x,y,z\in\mathbb{R}\right\}$$

$$=\left\{x\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+y\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}+z\begin{pmatrix}2a_1\\2a_2\\2a_3\end{pmatrix}\middle|\ x,y,z\in\mathbb{R}\right\}$$

$$=\left\langle\underbrace{\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}}_{v}\underbrace{\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}}_{w}\underbrace{\cancel{\begin{pmatrix}2a_1\\2a_2\\2a_3\end{pmatrix}}}_{2v}\right\rangle \Rightarrow \boxed{\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\quad\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}}$$

$$\boxed{f\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1x+b_1y+2a_1z\\a_2x+b_2y+2a_2z\\a_3x+b_3y+2a_3z\end{pmatrix}}$$

$$\Rightarrow f\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2x+2y+4z\\y\\-x+2y-2z\end{pmatrix}$$

NON è unica

(b) $\underbrace{A_{\mathcal{E}\mathcal{E}}(f)}_{\substack{\| \\ \ddot{A}}} = ?$

$A\begin{pmatrix}1\\0\\0\end{pmatrix} = f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$

$A\begin{pmatrix}0\\1\\0\end{pmatrix} = f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}$

$A\begin{pmatrix}0\\0\\1\end{pmatrix} = f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}4\\0\\-2\end{pmatrix}$

$$A_{\mathcal{E}\mathcal{E}}(f) = \begin{pmatrix}2 & 2 & 4\\ 0 & 1 & 0\\ -1 & 2 & -2\end{pmatrix}$$

$$f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x+2y+4z\\ y\\ -x+2y-2z\end{pmatrix}$$

(c) $$f^{-1}\left(\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}\right) = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; f\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; \begin{pmatrix}2x+2y+4z\\ y\\ -x+2y-2z\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

$$\begin{cases} 2x+2y+4z = 1\\ y = 1\\ -x+2y-2z = 1\end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 4 & 1\\ 0 & 1 & 0 & 1\\ -1 & 2 & -2 & 1\end{array}\right) \xrightarrow[\mathrm{I}+2\mathrm{III}]{-\mathrm{III}} \left(\begin{array}{ccc|c} 1 & -2 & 2 & -1\\ 0 & 1 & 0 & 1\\ 0 & 6 & 0 & 3\end{array}\right)$$

$$\xrightarrow[\mathrm{III}-6\mathrm{II}]{} \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & -2 & 2\\ 0 & 1 & 0\\ 0 & 0 & 0\end{matrix}}_{\tilde{A}} & \underbrace{\begin{matrix}-1\\ 1\\ -3\end{matrix}}_{\tilde{b}}\end{array}\right) \quad \text{R-C} \quad \nexists \text{ sol.}$$

$$f^{-1}\left(\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}\right) = \emptyset \iff \begin{pmatrix}1\\1\\1\end{pmatrix} \notin \operatorname{im} f$$

$$f^{-1}\left(\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}\right)=\emptyset \iff \begin{pmatrix}1\\1\\1\end{pmatrix}\notin \operatorname{im} f$$

$$f^{-1}\begin{pmatrix}0\\-1\\-3\end{pmatrix} \qquad \boxed{f\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\-1\\-3\end{pmatrix}}$$

$$\left(\begin{array}{ccc|c}2&2&4&0\\0&1&0&-1\\-1&2&-2&-3\end{array}\right)\longrightarrow \begin{array}{c}-\mathrm{III}\\ \\ \mathrm{I}+2\mathrm{III}\end{array}\left(\begin{array}{ccc|c}1&-2&2&3\\0&1&0&-1\\0&6&0&-6\end{array}\right)$$

$$\longrightarrow \begin{array}{c}\\ \\ \mathrm{III}-6\mathrm{II}\end{array}\left(\begin{array}{ccc|c}1&-2&2&3\\0&1&0&-1\\0&0&0&0\end{array}\right)$$

$$\begin{cases}x-2y+2z=3\\ y=-1\\ 0=0\end{cases} \Rightarrow \begin{cases}x=1-2z\\ y=-1\end{cases}$$

$$\Rightarrow f^{-1}\left(\left\{\begin{pmatrix}0\\-1\\-3\end{pmatrix}\right\}\right)=\left\{\begin{pmatrix}1-2z\\-1\\z\end{pmatrix}\middle|\, z\in\mathbb{R}\right\}$$

$$=\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}+z\underbrace{\begin{pmatrix}-2\\0\\1\end{pmatrix}}_{\text{generatore } \operatorname{Ker} f}\middle|\, z\in\mathbb{R}\right\} \quad \#$$