2020/04/16
giovedì 16 aprile 2020 10:06

F4/1

$\mathbb{R}^5$ $\quad W := \left\langle \underset{v_1}{\begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}2\\-1\\1\\1\\-1\end{pmatrix}}, \underset{v_3}{\begin{pmatrix}0\\-1\\-1\\1\\-3\end{pmatrix}}, \underset{v_4}{\begin{pmatrix}0\\1\\-1\\2\\3\end{pmatrix}}, \underset{v_5}{\begin{pmatrix}2\\-3\\3\\-3\\-7\end{pmatrix}} \right\rangle$

base di W? $\qquad W'$ t.c. $W \oplus W' = \mathbb{R}^5$

$\lambda_1, \dots, \lambda_5 \in \mathbb{R}$ $\qquad \sum_{i=1}^{5} \lambda_i v_i = 0$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 1 & 1 & -1 & -1 & 3 & 0 \\ 0 & 1 & 1 & 2 & -3 & 0 \\ 1 & -1 & -3 & 3 & -7 & 0 \end{array}\right) \longrightarrow \begin{array}{c} \\ \\ III-I \\ \\ V-I \end{array} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 0 & -1 & -1 & 2 & -5 & 0 \\ 0 & 1 & 1 & 2 & -3 & 0 \\ 0 & -3 & -3 & 3 & -9 & 0 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \\ \\ III-II \\ IV+II \\ V-3II \end{array} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow_{IV-3III} \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 2 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$W = \langle v_1, v_2, \cancel{v_3}, v_4, \cancel{v_5} \rangle = \langle v_1, v_2, v_4 \rangle$

$$\begin{cases} \lambda_1 + 2\lambda_2 + 2\lambda_5 = 0 \\ -\lambda_2 - \lambda_3 + \lambda_4 - 3\lambda_5 = 0 \\ \lambda_4 - 2\lambda_5 = 0 \end{cases}$$

$$\begin{cases} \lambda_1 = -2\lambda_2 - 2\lambda_5 = 2\lambda_3 \\ \lambda_2 = -\lambda_3 - \lambda_5 \\ \lambda_4 = 2\lambda_5 \end{cases}$$

$$\begin{cases} \lambda_1 = -2\lambda_2 - 2\lambda_3 = 2\lambda_3 \\ \lambda_2 = -\lambda_3 - \lambda_5 \\ \lambda_4 = 2\lambda_5 \end{cases}$$

$\lambda_4 - 2\lambda_5 = 0$

$\dim W = 3$ $\qquad W' \oplus W = \mathbb{R}^5 \qquad W \cap W' = \langle 0 \rangle$

$$\dim(W \oplus W') = \dim W + \dim W' - \dim(W \cap W')$$

$$\dim(\mathbb{R}^5) = 5 = 3 \quad \dim W' \quad 0 \quad \Rightarrow \dim W' = 2$$

$$W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \\ 3 \end{pmatrix} \right\rangle \qquad v_1 \quad v_2 \quad v_4$$

$$\begin{pmatrix} x \\ y \\ z \\ t \\ s \end{pmatrix} = a v_1 + b v_2 + c v_4$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & -1 & 1 & y \\ 1 & 1 & -1 & z \\ 0 & 1 & 2 & t \\ 1 & -1 & 3 & s \end{array}\right) \xrightarrow{\substack{III - I \\ V - I}} \left(\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & -1 & 1 & y \\ 0 & -1 & -1 & z-x \\ 0 & 1 & 2 & t \\ 0 & -3 & 3 & s-x \end{array}\right)$$

$$\xrightarrow{\substack{III - II \\ IV + II \\ V - 3II}} \left(\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & -1 & 1 & y \\ 0 & 0 & -2 & z-x-y \\ 0 & 0 & 3 & t+y \\ 0 & 0 & 0 & s-x-3y \end{array}\right) \xrightarrow{2IV + 3III} \left(\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & -1 & 1 & y \\ 0 & 0 & -2 & z-x-y \\ 0 & 0 & 0 & 2t+2y+3z-3x-3y \\ 0 & 0 & 0 & s-x-3y \end{array}\right)$$

$2t+2y+3z-3x-3y = 0$

$s-x-3y = 0$

$\tilde{A} \qquad \tilde{b}$

$\{ -3x - y + 3z + 2t = 0$

A b

$$\begin{cases} -3x - y + 3z + 2t = 0 \\ -x - 3y + s = 0 \end{cases}$$

$$W' = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$\cancel{\in}$ W $\quad$ $\cancel{\in}$ W

$$\begin{cases} -3 = 0 \\ -1 = 0 \end{cases} \quad 4$$

$$\begin{cases} 0 = 0 \\ 1 = 0 \end{cases} \quad 4$$

F4/3

$\mathbb{R}^4$

$$V = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -4 \\ -3 \end{pmatrix} \right\rangle$$

$v_1 \quad v_2 \quad v_3$

$\dim V = ?$

$V' \subseteq \mathbb{R}^4$ t.c. $V \subseteq V'$, $\dim V' = \dim V + 1$

eq. cartesiane di V'?

- $v \in V \iff v = a v_1 + b v_2 + c v_3 = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$

$$= \begin{pmatrix} a + b - c \\ 2a + b + c \\ a + 2b - 4c \\ b - 3c \end{pmatrix}$$

$$\begin{cases} a + b - c = x \\ a + b + c = y \\ a + 2b - 4c = z \\ b - 3c = t \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & x \\ 2 & 1 & 1 & y \\ 1 & 2 & -4 & z \\ 0 & 1 & -3 & t \end{array}\right) \longrightarrow \begin{array}{c} \\ IV \\ III - I \\ II - 2I \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -1 & x \\ 0 & 1 & -3 & t \\ 0 & 1 & -3 & z - x \\ 0 & -1 & 3 & y - 2x \end{array}\right)$$

$$\begin{array}{l} \longrightarrow \\ III-II \\ IV+II \end{array}\left(\begin{array}{ccc|c} 1 & 1 & -1 & x \\ 0 & 1 & -3 & t \\ 0 & 0 & 0 & z-x-t \\ 0 & 0 & 0 & y-2x+t \end{array}\right) \begin{array}{l} \\ \\ =0 \\ =0 \end{array}$$

$$V: \begin{cases} -x+z-t=0 \\ -2x+y+t=0 \end{cases} \qquad V=\langle v_1, v_2\rangle$$

$$\dim V=2$$

$$\dim(V')=3 \qquad V': \{-x+z-t=0$$

$V \leq V'$?

se $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in V \Rightarrow \begin{cases} -x+z-t=0 \\ -2x+y+t=0 \end{cases}$

$$\Rightarrow \{-x+z-t=0 \Rightarrow \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in V'$$

$$\Rightarrow V \subseteq V' \Rightarrow V \leq V'$$

$$V'=\left\{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \middle| -x+z-t=0\right\} = \left\{\begin{pmatrix} z-t \\ y \\ z \\ t \end{pmatrix} \middle| y,z,t \in \mathbb{R}\right\}$$

$$= \left\{ y\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle| y,z,t \in \mathbb{R}\right\}$$

$$= \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \dim V'=3$$

F4/S

$$\{ -a_{11} - a_{12} + a_{13} + 2a_{21} + a_{23} = 0 \quad \}$$

$$U = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \;\middle|\; \begin{cases} -a_{11} - a_{12} + a_{13} + 2a_{21} + a_{23} = 0 \\ -3a_{11} + a_{12} + a_{13} + 3a_{21} + a_{22} + 3a_{23} = 0 \\ a_{11} - a_{12} - a_{13} + a_{21} - a_{22} - a_{23} = 0 \\ 2a_{11} - 2a_{13} - a_{21} - a_{22} - 2a_{23} = 0 \end{cases} \right\}$$

Completare la base di U a tutto $M_{2\times 3}(\mathbb{R})$

$$\begin{array}{c} \begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{21} & a_{22} & a_{23} \end{array} \\ \left(\begin{array}{cccccc|c} -1 & -1 & 1 & 2 & 0 & 1 & 0 \\ -3 & 1 & 1 & 3 & 1 & 3 & 0 \\ 1 & -1 & -1 & 1 & -1 & -1 & 0 \\ 2 & 0 & -2 & -1 & -1 & -2 & 0 \end{array}\right) \end{array}$$

$$\rightarrow \begin{array}{c} \\ II-3I \\ III+I \\ IV+2I \end{array} \left(\begin{array}{cccccc|c} -1 & -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 4 & -2 & -3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 3 & -1 & 0 & 0 \\ 0 & -2 & 0 & 3 & -1 & 0 & 0 \end{array}\right)$$

$$\rightarrow \begin{array}{c} \\ III \\ II+2III \\ IV-III \end{array} \left(\begin{array}{cccccc|c} -1 & -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & -2 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{c} \\ I-\frac{II-III}{2} \\ \frac{II-III}{2} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 12 & 13 & 21 & 22 & 23 \end{array} \\ \left(\begin{array}{cccccc|c} -1 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$$\begin{cases} -a_{11} + 2a_{21} + a_{23} = 0 \\ -a_{12} + a_{13} = 0 \\ -2a_{13} + 3a_{21} - a_{22} = 0 \end{cases} \Rightarrow \begin{cases} a_{11} = 2a_{21} + a_{23} \\ a_{12} = a_{13} \\ a_{22} = -2a_{13} + 3a_{21} \end{cases}$$

$$-2a_{13}+3a_{21}-a_{22}=0 \qquad a_{22} = -2a_{13}+3a_{21}$$

$$U = \left\{ \begin{pmatrix} 2a_{21}+a_{23} & a_{13} & a_{13} \\ a_{21} & -2a_{13}+a_{21} & a_{23} \end{pmatrix} \middle| \; a_{13}, a_{21}, a_{23} \in \mathbb{R} \right\}$$

$$= \left\{ a_{13}\begin{pmatrix} 0 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} + a_{23}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \sim \in \mathbb{R} \right\}$$

$$U = \left\langle \begin{pmatrix} 0 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\dim U = 3 \qquad \dim\left(M_{2\times3}(\mathbb{R})\right) = 6$$

$$M_{2\times3}(\mathbb{R}) = \langle \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{U' \text{ t.c. } U \oplus U' = M_{2\times3}(\mathbb{R})} \rangle$$

$\boxed{F4/7}$

$$\mathbb{R}^{\le 3}[x] \qquad V = \langle \underbrace{-x^3+4x^2+x+3}_{v_1}, \underbrace{x^3+2x^2+x+1}_{v_2}, \underbrace{2x^3+x^2+x}_{v_3} \rangle$$

$\dim V = ?$, V' t.c. $V \subseteq V'$, $\dim V' = \dim V + 1$, eq. cart. V'

$\mathcal{B} = \{1, x, x^2, x^3\} \Rightarrow V = \left\langle \begin{pmatrix} 3 \\ 1 \\ 4 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\rangle$

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & 1 & 0 & d \\ 1 & 1 & 1 & c \\ 4 & 2 & 1 & b \\ -1 & 1 & 2 & a \end{array}\right) \to \begin{array}{c} II \\ I-3II \\ III-4II \\ IV+II \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & c \\ 0 & -2 & -3 & d-3c \\ 0 & -2 & -3 & b-4c \\ 0 & 2 & 3 & a+c \end{array}\right)$$

$$\to \begin{array}{c} \\ \\ III-II \\ IV+II \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & c \\ 0 & -2 & -3 & d-3c \\ 0 & 0 & 0 & b-4c-d+3c \\ 0 & 0 & 0 & a+c+d-3c \end{array}\right) \begin{array}{l} \\ \\ = b-c-d \\ = a-2c+d \end{array}$$

$\Rightarrow V = \langle v_1, v_2 \rangle$

$\Downarrow$

$\dim V = 2$

$$V : \begin{cases} b - c - d = 0 \\ a - 2c + d = 0 \end{cases}$$

$V' : \{ b - c - d = 0$

$V \leq V'$

$$V' = \left\{ \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \middle| \; b - c - d = 0 \right\} = \left\{ \begin{pmatrix} d \\ c \\ c+d \\ a \end{pmatrix} \middle| \; a, c, d \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \middle| \; \ldots \in \mathbb{R} \right\}$$

$$= \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \Rightarrow \dim V' = 3 = \dim V + 1$$

$$\Rightarrow V' = \langle 1, x+x^2, x+x^3 \rangle$$