Recap

$\mathbb{R}^3$, $\left\{ \begin{pmatrix} x\\ y\\ z\end{pmatrix} \;\middle|\; x+y=z \right\} = V$

criterio: - $\forall\, v_1, v_2 \in V \Rightarrow v_1+v_2 \in V$

- $\forall\, \lambda \in \mathbb{R}, \forall\, v \in V \Longrightarrow \lambda v \in V$

$v_1 = \begin{pmatrix} x_1\\ y_1\\ z_1\end{pmatrix}$ t.c. $x_1+y_1=z_1$, $v_2 = \begin{pmatrix} x_2\\ y_2\\ z_2\end{pmatrix}$ t.c. $x_2+y_2=z_2$

$v_1+v_2 = \begin{pmatrix} x_1+x_2\\ y_1+y_2\\ z_1+z_2\end{pmatrix} \overset{?}{\in} V$

$(x_1+x_2)+(y_1+y_2)=(z_1+z_2)$

$\underbrace{(x_1+y_1-z_1)}_{=0} + \underbrace{(x_2+y_2-z_2)}_{=0} = 0$ ✓

$\lambda,\ v = \begin{pmatrix} x\\ y\\ z\end{pmatrix} \in V$ $x+y=z$

$\lambda v = \begin{pmatrix} \lambda x\\ \lambda y\\ \lambda z\end{pmatrix} \overset{?}{\in} V$

$(\lambda x)+(\lambda y)=(\lambda z)$

$\lambda\underbrace{(x+y-z)}_{=0} = 0$ ✓

$\Rightarrow$ V è sp. vett. #

F 3/2

$\mathbb{R}^{\le 3}[x]$ se i seguenti sono sottosp.? calcolare la dim

(c) $W = \{ P(x) \in \mathbb{R}^{\le 3}[x] \mid P(x) = ax+bx^3,\ a,b \in \mathbb{R} \}$

(d) $V = \{ \lambda(x^2+1) \mid \lambda \in \mathbb{R} \}$

(c) • $P_i(x) = a_i x + b_i x^3$ $i=1,2$ $\left.\begin{matrix} P_1(x) = a_1x+b_1x^3 \\ P_2(x) = a_2x+b_2x^3 \end{matrix}\right\} \in W$

$P_1(x)+P_2(x) \in W$?

$\|$

$(a_1+a_2)x+(b_1+b_2)x^3 \in W$

$\|$

$0+(a_1+a_2)x+0\cdot x^2+(b_1+b_2)x^3$

- $\lambda \in \mathbb{R}$ $\quad \lambda \cdot p_1(x) \in ?\ W$

$$= \lambda a_1 x + \lambda b_1 x^3 \in W$$

- $P(x) \in W \Rightarrow P(x) = ax + bx^3 = a\cdot(x) + b\cdot(x^3)$

$W = \langle x, x^3 \rangle \Rightarrow \boxed{\dim W = 2}$

$$\lambda x + \mu x^3 = 0 = 0\cdot x + 0\cdot x^3$$

$$\Updownarrow$$

$$\begin{cases} \lambda = 0 \\ \mu = 0 \end{cases}$$

(d) $V = \{\lambda(x^2+1) \mid \lambda \in \mathbb{R}\} = \langle x^2+1 \rangle \Rightarrow V$ è sp. Vett.

$\Rightarrow \dim V = 1$

F 3/3

(f) $V = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \;\middle|\; \begin{cases} a_{11} - 2a_{12} = 0 \\ a_{11} - a_{12} + a_{22} = 0 \end{cases} \right\}$ $\quad K = \mathbb{R}$

V è sp. Vett? dim V?

- $\begin{array}{cccc} a_{11} & a_{12} & a_{21} & a_{22} \end{array}$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{II-I} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} a_{11} = 2a_{12} = -2a_{22} \\ a_{12} = -a_{22} \end{cases}$$

$A \in M_{2\times 2}(\mathbb{R})$ $\Leftrightarrow A = \begin{pmatrix} -2a_{22} & -a_{22} \\ a_{21} & a_{22} \end{pmatrix} = a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}$

$\cap$
V

$V = \left\{ a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} \;\middle|\; a_{21}, a_{22} \in \mathbb{R} \right\}$

$$= \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} \right\rangle \Rightarrow V \text{ è sp. vett.}$$

$$\lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mu \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow \begin{cases} -2\mu = 0 \\ -\mu = 0 \\ \lambda = 0 \\ \mu = 0 \end{cases} \Leftrightarrow \lambda = \mu = 0$$

$\Rightarrow \dim V = 2$

F3/5 determinare $W_1, W_2, W_1 \cap W_2, W_1 + W_2$ e dimensioni

(e) $W_1 = \langle \underset{v_1}{1+2x^2}, \underset{v_2}{x+x^3} \rangle \qquad W_2 = \langle \underset{v_3}{1+x^2}, \underset{v_4}{x^2} \rangle$

$$\lambda(1+2x^2) + \mu(x+x^3) = 0 = 0 + 0x + 0 \cdot x^2 + 0 \cdot x^3$$

$$\lambda + \mu x + 2\lambda x^2 + \mu x^3 \qquad \begin{cases} \lambda = 0 \\ \mu = 0 \\ 2\lambda = 0 \\ \mu = 0 \end{cases} \Leftrightarrow \lambda = \mu = 0$$

$\Rightarrow \dim W_1 = 2$

$$\lambda(1+x^2) + \mu(x^2) = 0 \Leftrightarrow \begin{cases} \lambda = 0 \\ \lambda + \mu = 0 \end{cases} \Leftrightarrow \lambda = \mu = 0$$

$\Rightarrow \dim W_2 = 2$

$$W_1 + W_2 = \langle \underbrace{\cancel{1+2x^2}, x+x^3}_{W_1}, \underbrace{1+x^2, x^2}_{W_2} \rangle = \langle x+x^3, 1+x^2, x^2 \rangle$$

W_1

$\cup$

$v_1 = 1+2x^2 = 1+x^2+x^2 = \underbrace{v_3+v_4}_{\in W_2}$

$$\lambda_1(x+x^3) + \lambda_2(1+x^2) + \lambda_3 x^2 = 0$$

$$\lambda_2 + \lambda_1 x + (\lambda_2+\lambda_3)x^2 + \lambda_1 x^3 = 0 \Leftrightarrow \begin{cases} \lambda_2 = 0 \\ \lambda_1 = 0 \\ \lambda_2+\lambda_3 = 0 \\ \lambda_1 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$\dim(W_1 + W_2) = 3$

$1\lambda_1 = 0$

$\dim(W_1+W_2)=3$

$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1\cap W_2)$

$3 = 2 + 2 - \dim(W_1\cap W_2)$

$\Rightarrow \dim(W_1\cap W_2) = 1$

$\Rightarrow v_1 \in W_1\cap W_2 \Rightarrow W_1\cap W_2 = \langle v_1\rangle = \langle 1+2x^2\rangle$

F3/6 determinare $W_1, W_2, W_1+W_2, W_1\cap W_2$

$$W_1 = \left\langle \underset{v_1}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0\end{pmatrix}} \right\rangle \qquad W_2 = \left\langle \underset{v_2}{\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0\end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0\end{pmatrix}} \right\rangle$$

$\dim(W_1)=1$

$\dim(W_2)=2$

$$\boxed{\lambda v_2 + \mu v_3 = 0}$$

$$\begin{pmatrix} \lambda & 0 & 2\lambda+\mu \\ \lambda+\mu & 2\lambda+\mu & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0\end{pmatrix}$$

$$W_1+W_2 = \left\langle \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0\end{pmatrix}}, \underbrace{\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0\end{pmatrix}} \right\rangle$$

$$\begin{cases} \lambda = 0 \\ 2\lambda+\mu = 0 \\ \lambda+\mu = 0 \\ 2\lambda+\mu = 0 \end{cases} \Leftrightarrow \lambda = 0 = \mu$$

$$\boxed{\lambda v_2 + \mu v_3 = v_1}$$

$$\begin{pmatrix} \lambda & 0 & 2\lambda+\mu \\ \lambda+\mu & 2\lambda+\mu & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0\end{pmatrix} \Leftrightarrow \begin{cases} \lambda = 0 \\ 2\lambda+\mu = 1 \\ \lambda+\mu = 1 \\ 2\lambda+\mu = 2 \end{cases}$$

$1 = 2\lambda+\mu = 2$

$\lightning$

$\dim(W_1+W_2) = 3$

$\dim(W_1\cap W_2) = \dim W_1 + \dim W_2 - \dim(W_1+W_2)$

$= 1 + 2 - 3 = 0$

$W_1 \cap W_2 = \langle 0 \rangle$

F3/8

$\mathbb{R}^4$ completare/estrarre base degli sp. vett.

ⓒ $S_1 = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right\rangle$ with $v_1\ v_2\ v_3\ v_4$

ⓓ $S_2 = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\3\\0\\0\end{pmatrix} \right\rangle$ with $v_1\ v_2\ v_3\ v_4\ v_5$

ⓒ

$v_4 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\\1\end{pmatrix} = v_1 + v_2$

$S_1 = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \right\rangle$

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$

$\Updownarrow$

$$\begin{pmatrix}\lambda_1 + \lambda_3\\ \lambda_2 + \lambda_3\\ \lambda_2 + \lambda_3\\ \lambda_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Leftrightarrow \begin{cases}\lambda_1 + \lambda_3 = 0 & \Rightarrow \lambda_1 = 0\\ \lambda_2 + \lambda_3 = 0\\ \lambda_2 + \lambda_3 = 0 & \Rightarrow \lambda_3 = 0\\ \lambda_2 = 0\end{cases}$$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$ ✓ $\Rightarrow \dim S_1 = 3$

$$\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \underbrace{v_2 - v_3 + v_1}_{\in S_1}$$

$\left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \underset{e_3}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}} \right\rangle = \mathbb{R}^4$?

$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\1\\1\end{pmatrix} \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \begin{pmatrix}x\\y\\z\\t\end{pmatrix}$

$$\boxed{\lambda_1 \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\\1\\1\end{pmatrix} + \lambda_3 \begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \lambda_4 \begin{pmatrix}x\\y\\z\\t\end{pmatrix} = 0}$$

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 e_3 = 0$

$$\begin{pmatrix}\lambda_1 + \lambda_3\\ \lambda_2 + \lambda_3\\ \lambda_2 + \lambda_3 + \lambda_4\\ \lambda_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Longleftrightarrow \begin{cases}\lambda_1 + \lambda_3 = 0\\ \lambda_2 + \lambda_3 = 0\\ \lambda_2 + \lambda_3 + \lambda_4 = 0\\ \lambda_2 = 0\end{cases} \quad \begin{cases}\lambda_1 = 0\\ \lambda_3 = 0\\ \lambda_4 = 0\\ \lambda_2 = 0\end{cases}$$ ✓

(d) $S_2 = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\3\\0\\0\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\\0\end{pmatrix} \right\rangle = \mathbb{R}^4$

$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \qquad\qquad v_1 \quad v_2 \quad v_3 \quad v_5$

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = v_5$$

$$\begin{cases} \lambda_1 + \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 3 \\ \lambda_2 + \lambda_3 = 0 \\ \lambda_2 = 0 \end{cases} \quad ↯ \quad 3 = \lambda_2 + \lambda_3 = 0 \quad ↯$$

F 3/12 $\{v_1, \dots, v_n\}$ è base di V. Dimostrare che sono basi:

(e) $\mathcal{B} = \{\underbrace{v_1, v_1 + 2v_2, \dots, v_1 + 2v_n}_{n}\}$ (f) $\mathcal{B}' = \{v_1, v_1 + v_2, \dots, v_1 + \dots + v_n\}$

(e) $\lambda_1, \dots, \lambda_n$ t.c. $\lambda_1 v_1 + \lambda_2(v_1 + 2v_2) + \dots + \lambda_n(v_1 + 2v_n) = 0$

$$\| $$

$$(\lambda_1 + \dots + \lambda_n) v_1 + 2\lambda_2 v_2 + 2\lambda_3 v_3 + \dots + 2\lambda_n v_n$$

$$\begin{cases} \lambda_1 + \dots + \lambda_n = 0 \\ \cancel{2}\lambda_2 = 0 \\ \cancel{2}\lambda_3 = 0 \\ \vdots \\ \cancel{2}\lambda_n = 0 \end{cases} \Rightarrow \lambda_1 + \underbrace{\lambda_2 + \dots + \lambda_n}_{\overset{\|}{0}} = 0 \Rightarrow \lambda_1 = 0$$

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

(F) $\lambda_1, \dots, \lambda_n$ t.c. $\lambda_1 v_1 + \lambda_2(v_1 + v_2) + \lambda_3(v_1 + v_2 + v_3) + \dots + \lambda_n(v_1 + \dots + v_n) = 0$

$$\|$$

$$(\lambda_1 + \lambda_2 + \dots + \lambda_n) v_1 + (\lambda_2 + \dots + \lambda_n) v_2 + (\lambda_3 + \dots + \lambda_n) v_3 + \dots + \lambda_n v_n$$

$$\begin{cases} \lambda_1 + \lambda_2 + \dots + \lambda_n = 0 \\ \lambda_2 + \dots + \lambda_n = 0 \\ \lambda_3 + \dots + \lambda_n = 0 \\ \vdots \\ \lambda_n = 0 \end{cases}$$

è vero che $\lambda_i = 0 \quad \forall i \in \{1, \dots, n\}$?

$\lambda_n = 0$

$\lambda_i = 0$?

$$\begin{cases} \lambda_i + \lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_n = 0 & \text{I} \\ \lambda_{i+1} + \lambda_{i+2} + \dots + \lambda_n = 0 & \text{II} \end{cases}$$

$$\lambda_i + 0 + 0 + \dots + 0 = 0 \qquad \text{I} - \text{II}$$

$$\Updownarrow$$

$$\lambda_i = 0 \quad \forall i \in \{1, \dots, n\}$$

es casa F3/11