2020/03/24

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U, W sp. Vett. $\Rightarrow$ ~~$U \cup W$ è sp. vett.?~~

$U+W = \{ u+w \mid u \in U, w \in W \}$ ✓

es 1

$U = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \qquad W = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \qquad \mathbb{R}^3$

$v_1 \quad v_2 \qquad v_3 \quad v_4$

• $U+W = \mathbb{R}^3$?

$U \cap W = \langle 0 \rangle$?

$U \subseteq \mathbb{R}^3, W \subseteq \mathbb{R}^3 \Rightarrow U+W \subseteq \mathbb{R}^3$

$\supseteq$

$U+W = \langle v_1, v_2, v_3, v_4 \rangle$

$v \in \mathbb{R}^3 \Rightarrow v \in U+W$

$$\boxed{\begin{pmatrix} a \\ b \\ c \end{pmatrix} = v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4}$$

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a \\ 2\lambda_1 + \lambda_3 = b \\ \lambda_1 + \lambda_2 - \lambda_4 = c \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 2 & 0 & 1 & 0 & b \\ 1 & 1 & 0 & -1 & c \end{array}\right) \longrightarrow \begin{array}{c} \\ II-2I \\ III-I \end{array} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & -2 & -1 & -2 & b-2a \\ 0 & 0 & -1 & -2 & c-a \end{array}\right)$$

per R-C ✓

$U+W=\mathbb{R}^3$

$U \cap W$

$\dim(W+U)=\dim W+\dim U-\dim(U\cap W)$

$3 = 2 + 2 - \dim(U\cap W)$

$\Rightarrow \dim(U\cap W) = 4-3 = 1$

$$\boxed{\lambda_1\begin{pmatrix}1\\2\\1\end{pmatrix}+\lambda_2\begin{pmatrix}1\\0\\1\end{pmatrix}=0} \Rightarrow \boxed{\lambda_1=\lambda_2=0\,?}$$

$$\begin{cases}\lambda_1+\lambda_2=0\\ \lambda_1=0\\ -\lambda_2=0\end{cases} \longrightarrow \begin{cases}\lambda_1=0\\ \lambda_2=0\end{cases} \quad ✓$$

$v \in U\cap W$

$$\underbrace{\lambda_1 v_1+\lambda_2 v_2}_{\in U} = v = \underbrace{\lambda_3 v_3+\lambda_4 v_4}_{\in W}$$

$v=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\begin{pmatrix}\lambda_1+\lambda_2\\2\lambda_1\\\lambda_1+\lambda_2\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}\lambda_3+\lambda_4\\\lambda_3\\-\lambda_4\end{pmatrix}$$

$$\left(\begin{array}{cc|c}1&1&a\\2&0&b\\1&1&c\end{array}\right)\longrightarrow \begin{array}{c}II/2\\ I-II/2\\ III-I\end{array}\left(\begin{array}{cc|c}1&0&b/2\\0&1&a-b/2\\0&0&\boxed{c-a}\end{array}\right)$$

$c-a=0 \quad \boxed{c=a}$

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 1 & 0 & b \\ 0 & -1 & a \end{array}\right)$$

$$\begin{array}{c} \mathrm{II} \\ \mathrm{III} \\ \mathrm{I}-\mathrm{II}+\mathrm{III} \end{array} \left(\begin{array}{cc|c} 1 & 0 & b \\ 0 & -1 & a \\ 0 & 0 & \underbrace{a-b+a}_{2a-b} \end{array}\right)$$

$2a-b=0$

$b=2a$

$$\begin{pmatrix} a \\ 2a \\ a \end{pmatrix} = a \cdot \boxed{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$$

$$U \cap W = \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$$

es 2

$$\begin{cases} 2x_1 - x_2 + x_3 - x_4 = 2 \\ 4x_1 - 3x_2 - x_3 + 3x_4 = 1 \\ 2x_1 + x_2 + 13x_3 + x_4 = 17 \end{cases}$$

- ha soluzioni? $\mathbb{R}^4$

  chi sono?

- chi sono le soluzioni del omogeneo associato?

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 2 \\ 4 & -3 & -1 & 3 & 1 \\ 2 & 1 & 13 & 1 & 17 \end{array}\right) \longrightarrow \begin{array}{c} \\ \mathrm{II}-2\mathrm{I} \\ \mathrm{III}-\mathrm{I} \end{array} \left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 2 \\ 0 & -1 & -3 & 5 & -3 \\ 0 & 2 & 12 & 2 & 15 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \\ \\ \frac{\mathrm{III}+2\mathrm{II}}{3} \end{array} \left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 2 \\ 0 & -1 & -3 & 5 & -3 \\ 0 & 0 & 2 & 4 & 3 \end{array}\right)$$

$$\begin{cases} 2x_1 - x_2 + x_3 - x_4 = 2 \\ \quad + x_2 + 3x_3 - 5x_4 = +3 \\ \quad 2x_3 + 4x_4 = 3 \end{cases} \Rightarrow \boxed{\begin{cases} 2x_1 = 2 + x_2 - x_3 + x_4 \\ x_2 = 3 - 3x_3 + 5x_4 \\ x_3 = \frac{3}{2} - 2x_4 \end{cases}}$$

$\Rightarrow x_2 = 3 - \frac{9}{2} + 6x_4 + 5x_4$

$= -\frac{3}{2} + 11x_4$

$2x_1 = -\frac{1}{2} + 7x_4$

$$\boxed{\begin{cases} x_1 = -\frac{1}{2} + 7x_4 \\ x_2 = -\frac{3}{2} + 11x_4 \\ x_3 = \frac{3}{2} - 2x_4 \end{cases}}$$

$$\Rightarrow 2x_1 = 2 - \frac{3}{2} + 11x_4 - \frac{3}{2} + 2x_4 + x_4$$

$$= -1 + 14x_4$$

$$\Rightarrow x_1 = -\frac{1}{2} + 7x_4$$

$$S = \left\{ \begin{pmatrix} -\frac{1}{2} + 7x_4 \\ -\frac{3}{2} + 11x_4 \\ \frac{3}{2} - 2x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x_4 \in \mathbb{R} \right\}$$

$$= \begin{pmatrix} -1/2 \\ -3/2 \\ 3/2 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 7 \\ 11 \\ -2 \\ 1 \end{pmatrix} \right\rangle$$

- $$\begin{cases} 2x_1 - x_2 + x_3 - x_4 = 0 \\ 4x_1 - 3x_2 - x_3 + 3x_4 = 0 \\ 2x_1 + x_2 + 13x_3 + x_4 = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 4 & -3 & -1 & 3 & 0 \\ 2 & 1 & 13 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 5 & 0 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right)$$

$$S_0 = \left\{ \begin{pmatrix} 7a \\ 11a \\ -2a \\ a \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; a \in \mathbb{R} \right\}$$

$$= \left\langle \begin{pmatrix} 7 \\ 11 \\ -2 \\ 1 \end{pmatrix} \right\rangle$$

es 3

$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\qquad \mathbb{R}^3$

$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\mathbb{R}$

- $\{v_1, v_2, v_3\}$ Forma una base di $\mathbb{R}^3$

① Lin. ind.

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \Rightarrow \begin{pmatrix} 0\lambda_1 + \lambda_2 + 0\cdot\lambda_3 \\ 1\cdot\lambda_1 + 0\cdot\lambda_2 + 0\cdot\lambda_3 \\ 0\cdot\lambda_1 + \lambda_2 + \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \lambda_2 = 0 \\ \lambda_1 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases} \quad \checkmark$$

- $v \in \mathbb{R}^3, \quad v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \exists\, \mu_1, \mu_2, \mu_3 \in \mathbb{R}$ t.c.

$$\mu_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{cases} \mu_2 = a \\ \mu_1 = b \\ \mu_2 + \mu_3 = c \end{cases} \Rightarrow \begin{cases} \mu_1 = b \\ \mu_2 = a \\ \mu_3 = c - a \end{cases}$$

- $\underset{w_1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \underset{w_2}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \underset{w_3}{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}$

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \lambda_1 = 0 \\ \cancel{2\lambda_1} + \lambda_2 = 0 \\ \cancel{3\lambda_1} + \cancel{\lambda_2} - \lambda_3 = 0 \end{cases}$$

es 4

$\mathbb{R}^3$

$$\begin{cases} -3x_1 - 2x_2 - x_3 = 1 \\ 7x_1 + 3x_2 + 3x_3 = 2 \\ 13x_1 + 2x_2 + 7x_3 = 3 \\ 2x_1 - 2x_2 - 2x_3 = 4 \end{cases}$$

NO sol.

SI sol.

$$\left(\begin{array}{ccc|c} -3 & -2 & -1 & 1 \\ 7 & 3 & 3 & 2 \\ 13 & 2 & 7 & 3 \\ 2 & -2 & -2 & 4 \end{array}\right) \longrightarrow \begin{array}{c} IV/2 \\ II-7IV/2 \\ III-13IV/2 \\ I+3IV/2 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 10 & 10 & -12 \\ 0 & 15 & 20 & -23 \\ 0 & -5 & -4 & 7 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \\ II/2 \\ III+3IV \\ IV \end{array} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 5 & 5 & -6 \\ 0 & 0 & 8 & -2 \\ 0 & -5 & -4 & 7 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \\ \\ III/2 \\ IV+II \end{array} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 5 & 5 & -6 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \begin{array}{c} \\ \\ IV \\ III-4IV \end{array} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 5 & 5 & -6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right)$$

$$\begin{cases} \\ \\ \\ 0 = -5 \end{cases}$$

↯

$$\begin{cases} -3x_1 - 2x_2 - x_3 = 0 \\ 7x_1 + 3x_2 + 3x_3 = 0 \\ 13x_1 + 2x_2 + 7x_3 = 0 \\ 2x_1 - 2x_2 - 2x_3 = 0 \end{cases}$$

$\boxed{\text{es casa}}$

$\mathbb{R}^6$

$$\begin{cases} x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 + 3x_6 = a \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 + 2x_3 - x_5 + 4x_6 = b \\ -2x_1 - 2x_2 - 3x_3 - 5x_4 - 6x_5 - 2x_6 = c \end{cases}$$

- soluzioni se $a = 3, b = 4, c = 1$
- per quali a, b, c il sistema NON ha soluzione
- per quali a, b, c il sistema ha soluzione unica