TUTORATO 25/05

Ex. 1

$\mathbb{A}^3(\mathbb{R})$

$$r : \begin{cases} x - y + z = 2 \\ 3x - y - z = 2 \end{cases} \qquad s = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

(a)

$$r : \begin{cases} z = -x + y + 2 = -x + 2x \cancel{-2} \cancel{+2} = x \\ 3x - y + x - y - 2 = 2, \quad 4x - 2y = 4, \quad y = 2x - 2 \end{cases}$$

$$\begin{cases} z = x \\ y = 2x - 2 \end{cases}$$

$$r = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle$$

FORMA PARAMETRICA DI r:

$$r : \begin{cases} x = t \\ y = -2 + 2t \\ z = t \end{cases}, \quad t \in \mathbb{R}$$

(b) FORMA CARTESIANA DI s.

$$s = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

$$\begin{cases} x = 1 + k \\ y = -1 + k \\ z = 2 \end{cases} \rightarrow \quad k = x - 1$$

$$s: \begin{cases} y = -1 + x - 1 \\ z = 2 \end{cases} \qquad \begin{cases} x - y = 2 \\ z = 2 \end{cases}$$

(c)

$$rk \begin{pmatrix} 1 & -1 & 1 \\ 3 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 \qquad (A|b)$$

$$\begin{cases} a - b + \cancel{c} = 0 \\ 3a - b - \cancel{c} = 0 \rightarrow 2a = 0 \\ a - b = 0 \rightarrow b = a \\ c = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}.$$

$$rk \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & -1 & -1 & 2 \\ 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) = 4$$

$$\begin{cases} a-b+\cancel{c}+\cancel{2d}=0 \\ 3a-b-c+2d=0 \\ \cancel{a}-\cancel{b}+2d=0 \\ c+2d=0 \longrightarrow 2d=-c \end{cases}$$

$$\begin{cases} a=b & \longrightarrow b=0 \\ 3a-a-2c=0 & \longrightarrow a=0 \\ -c=0 \longrightarrow c=0 \\ d=0 \end{cases}$$

$\Rightarrow r$ e s SGHEMBE.

(d) $P=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\pi /\!/ r$, $\pi /\!/ s$, $P\in\pi$.

$$\pi = \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle$$

$$\pi : \begin{cases} x=1+t+s \\ y=1+2t+s \\ z=1+t \end{cases} \qquad t,s\in\mathbb{R}$$

$\hookrightarrow t=z-1$

$x = z + s \longrightarrow s = x - z$

$y = 1 + 2z - 2 + s$

$y = 1 + 2z - 2 + x - t$

$\Longrightarrow \pi : \boxed{x - y + z = 1}$.

Ex. 4.

$\mathbb{A}^3(\mathbb{R})$

$P_1 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(a) ? s (retta) $P_1 \in s$, $P_2 \in s$.

$s = P_1 + \langle P_1P_2 \rangle$

$P_1P_2 = P_2 - P_1 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

$\Longrightarrow s = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\rangle$

$\longrightarrow$ PARAMETRICA:

$s: \begin{cases} x = 2 - t \\ y = -t \\ z = 2 \end{cases}$

$\rightarrow$ CARTESIANA:

$$s: \begin{cases} x - y = 2 \\ z = 2 \end{cases}$$

(b) ?π (piano) $P_2 \in \pi$, $P_3 \in \pi$, $P_4 \in \pi$.

$\pi: ax + by + cz = d$

$$\begin{aligned} &\rightarrow P_2 \in \pi: \\ &\rightarrow P_3 \in \pi: \\ &\rightarrow P_4 \in \pi: \end{aligned} \begin{cases} a - b + 2c = d \\ 2a + 2b + 2c = d \\ a + c = d \end{cases}$$

$$\begin{cases} \cancel{a} - b + 2c = \cancel{a} + c \quad \rightarrow \quad c = b \\ 2a + 2b + 2c = a + c \\ d = a + c \end{cases}$$

$$\begin{cases} c = b \\ a + 3c = 0 \rightarrow a = -3c \\ d = -2c \end{cases}$$

$$\rightarrow c = 1 \implies \begin{cases} a = -3 \\ b = 1 \\ d = -2 \end{cases}$$

$\pi: \boxed{-3x + y + z = -2}$ EQ. CARTESIANA

$z = 3x - y - 2$

$$\pi = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

$$\pi : \begin{cases} x = t \\ y = s \\ z = -2 + 3t - s \end{cases} \quad , t, s \in \mathbb{R} .$$

EQ. PARAMETRICA.

(c) POSIZIONE RECIPROCA di π e s.

$$rk \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -3 & 1 & 1 \end{pmatrix} = 3$$

$$- \begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} = 2 \neq 0$$

$$rk \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ -3 & 1 & 1 & -2 \end{array} \right) = 3$$

$\Longrightarrow$ s e π INCIDENTI.

$$s \cap \pi : \begin{cases} x - y = 2 \\ z = 2 \\ -3x + y + z = -2 \end{cases}$$

$y = x - 2$

$z = 2$

$-3x + x \cancel{-2} \cancel{+2} = -2$

$-2x = -2 \implies x = 1$

$$\implies \begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases} \qquad P = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = s \cap \pi .$$

Ex.

$$r : \begin{cases} x + 3y + 1 = 0 \\ x + 3y + z - 2 = 0 \end{cases} \qquad s : \begin{cases} x - 2z - 7 = 0 \\ y - z - 4 = 0 \end{cases}$$

verificare che r e s siano SGHEMBE.

$$rk \left(\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 1 & 3 & 1 & 2 \\ 1 & 0 & -2 & 7 \\ 0 & 1 & -1 & 4 \end{array} \right)$$

$$\operatorname{rk}\begin{pmatrix} 1 & 3 & 0 \\ 1 & 3 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} = 3$$

$$\begin{array}{cl} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -2 \\ 0 & 1 & 1 \end{pmatrix} & \begin{array}{l} \mathrm{I} \\ \overline{\mathrm{II}} - \mathrm{I} \\ \overline{\mathrm{III}} - \mathrm{I} \\ \overline{\mathrm{IV}} \end{array} \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{array}{cl} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} & \begin{array}{l} \mathrm{I} \\ \overline{\mathrm{III}} \\ \mathrm{II} - 3\overline{\mathrm{III}} \\ \end{array} \end{array}$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{rk}\left(\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 1 & 3 & -1 & 2 \\ 1 & 0 & -2 & 7 \\ 0 & 1 & -1 & 4 \end{array}\right) = 4$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & -3 & -2 & 8 \\ 0 & 1 & -1 & 4 \end{array}\right) \begin{array}{l} I \\ II - I \\ III - I \\ IV \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -5 & 20 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} I \\ IV \\ III + 3IV \\ II \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 35 \end{array}\right) \begin{array}{l} I \\ II \\ IV \\ III + 5IV \end{array}$$

$\Rightarrow r$ e s <u>SGHEMBE</u>.

- retta ℓ incidente r e s
  $\ell \perp r, \ell \perp s$.
- $\text{dist}(r, s) = ?$
- $R \in r$, $S \in s$ di minima distanza

$$r: \begin{cases} x + 3y + 1 = 0 \\ x + 3y + z - 2 = 0 \end{cases} \qquad s: \begin{cases} x - 2z - 7 = 0 \\ y - z - 4 = 0 \end{cases}$$

$A, B \in r$

$$A = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$v_r = B - A = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \longrightarrow r = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \left\langle \underbrace{\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}}_{v_r} \right\rangle$$

$C, D \in s$

$$C = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \quad D = \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$$

$$v_s = C - D = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \longrightarrow s = \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} + \left\langle \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}}_{v_s} \right\rangle$$

$$P \in r, \quad P = \begin{pmatrix} -1 + 3\lambda \\ -\lambda \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$Q \in s, \quad Q = \begin{pmatrix} -1 + 2\mu \\ \mu \\ -4 + \mu \end{pmatrix}, \mu \in \mathbb{R}$$

$$w = P - Q = \begin{pmatrix} 3\lambda - 2\mu \\ -\lambda - \mu \\ 7 - \mu \end{pmatrix} \quad v_r = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \quad v_s = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} w \cdot v_r = 0 \\ w \cdot v_s = 0 \end{cases}$$

$$(3\lambda - 2\mu, -\lambda - \mu, 7 - \mu) \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$9\lambda - 6\mu + \lambda + \mu = 0$$

$$10\lambda - 5\mu = 0$$

$$(3\lambda - 2\mu, -\lambda - \mu, 7 - \mu) \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$6\lambda - 4\mu - \lambda - \mu + 7 - \mu = 0$$

$$5\lambda - 6\mu + 7 = 0$$

$$\begin{cases} 10\lambda - 5\mu = 0 \longrightarrow \mu = 2\lambda = 2 \\ 5\lambda - 6\mu + 7 = 0 \end{cases}$$

$$5\lambda - 12\lambda + 7 = 0 \longrightarrow \lambda = 1$$

$\boxed{\lambda = 1}$ $\boxed{\mu = 2}$

$$P = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad Q = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

$$r = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \langle Q - P \rangle = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} \right\rangle$$

$$r : \begin{cases} x = 2 + t \\ y = -1 + 3t \\ z = 3 - 5t \end{cases}, \quad t \in \mathbb{R}$$

$$t = x - 2$$

$$r : \begin{cases} 3x - 6 - 1 = y \\ z = 3 - 5x + 10 \end{cases} \longrightarrow \begin{cases} 3x - y = 7 \\ 5x + z = 13 \end{cases}$$

$$R = P = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \in r$$

$$S = Q = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \in s$$

$\Rightarrow \text{dist}(r, s) = \text{dist}(P, Q) = \|w\| =$

$= \left\| \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix} \right\| = \sqrt{1+9+25} = \sqrt{35}.$

<u>EX.</u>

$$r : \begin{cases} x - 2y - 3 = 0 \\ 2x + y + z + 1 = 0 \end{cases} \qquad P = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

(a) $? \; \pi \supset r, \; P \in \pi$

FASCIO DI PIANI DI ASSE r:

$\lambda(x - 2y - 3) + \mu(2x + y + z + 1) = 0$

$\to P \in \pi$

$\lambda(1 - 6 - 3) + \mu(\cancel{2} + 3 - \cancel{2} + 1) = 0$

$-8\lambda + 4\mu = 0$

$\Rightarrow \boxed{\mu = 2\lambda}$

$\lambda = 1, \; \mu = 2$

$\pi: x - \cancel{2y} - 3 + 4x + \cancel{2y} + 2z + 2 = 0$

$\boxed{5x + 2z = 1} \qquad n_\pi = (5, 0, 2)$

(b) retta s?

$\rightarrow P \in s$

$\rightarrow s \perp r$

$\rightarrow s \subset \pi$.

$$r: \begin{cases} x - 2y - 3 = 0 \\ 2x + y + z + 1 = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \in r \qquad B = \begin{pmatrix} 3 \\ 0 \\ -7 \end{pmatrix} \in r$$

$$v_r = B - A = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$$

$v_s \rightarrow$ VETTORE DIRETTORE DI s.

$v_s \perp v_r, \; v_s \perp n_\pi$

$$v_s = v_r \times n_\pi = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 1 & -5 \\ 5 & 0 & 2 \end{vmatrix} = (2, -29, -5)$$

$$s = P + \langle v_s \rangle$$

$$s: \begin{cases} x = 1 + 2t \\ y = 3 - 29t \\ z = -2 - 5t \end{cases}, \quad t \in \mathbb{R}$$

(c) $R = r \cap s$

$$R = r \cap s: \begin{cases} x - 2y - 3 = 0 \\ 2x + y + z + 1 = 0 \\ x = 1 + 2t \\ y = 3 - 29t \\ z = -2 - 5t \end{cases}$$

$$t = \frac{1}{2}x - \frac{1}{2}$$

$$y = 3 - \frac{29}{2}x + \frac{29}{2} = \frac{35}{2} - \frac{29}{2}x$$

$$z = -2 - \frac{5}{2}x + \frac{5}{2} = \frac{1}{2} - \frac{5}{2}x$$

$$s: \begin{cases} 29x + 2y = 35 \\ 5x + 2z = 1 \end{cases}$$

$$\begin{cases} x - 2y - 3 = 0 \\ 2x + y + z + 1 = 0 \\ 29x + 2y = 35 \\ 5x + 2z = 1 \end{cases}$$

$$x = 2y + 3 = -\frac{26}{15} + 3 = \frac{19}{15}$$

$$\to 4y + 6 + y + z + 1 = 0$$

$$58y + 87 + 2y = 35 \longrightarrow 60y = -52$$

$$\Longrightarrow y = -\frac{52}{60} = -\frac{13}{15}$$

$$z = -5y - 7 = \frac{13}{3} - 7 = -\frac{8}{3}$$

$$5x + 2z = 1$$

$$\frac{19}{3} - \frac{16}{3} = 1 \qquad \frac{3}{3} = 1 \ \checkmark.$$

$$R = r \cap s = \left(\frac{19}{15}, -\frac{13}{15}, -\frac{8}{3}\right) \qquad P = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$\mathrm{dist}(P, r) =$$

$$= \mathrm{dist}(P, R) = \|R - P\| = \left\| \begin{pmatrix} 4/15 \\ -58/15 \\ -2/3 \end{pmatrix} \right\| =$$

$$= \sqrt{\frac{16}{225} + \frac{58^2}{225} + \frac{4}{9}} = \frac{2}{15}\sqrt{870}.$$