TUTORATO 18/05.

Ex. 2.

$a \in \mathbb{R}$

$$U_a = \left\langle \underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{=u_1}, \underbrace{\begin{pmatrix}1+a\\-a\\a-1\end{pmatrix}}_{=u_{2,a}} \right\rangle$$

$$v = \begin{pmatrix}2\\-1\\3\end{pmatrix}$$

(2) $\boxed{a=1}$.

$$U_1 = \left\langle \underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{=u_1}, \underbrace{\begin{pmatrix}2\\-1\\0\end{pmatrix}}_{=u_2} \right\rangle$$

$p_{U_1}(v) = ?$

$p_{U_1}(v) = v - p_{U_1^\perp}(v)$

$$U_1^\perp = \left\langle \underbrace{\begin{pmatrix}x\\y\\z\end{pmatrix}}_{=w_1} \right\rangle = \left\langle \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\rangle$$

$$\begin{cases} w_1 \cdot u_1 = 0 \longrightarrow \\ w_1 \cdot u_2 = 0 \longrightarrow \end{cases} \begin{cases} x + z = 0 \longrightarrow z = -x \\ 2x - y = 0 \longrightarrow y = 2x \end{cases}$$

BASE ORTONORMALE di $U_1^\perp$ =

$$\left\{ v_1 = \frac{w_1}{\|w_1\|} \right\}$$

$$\|w_1\| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$P_{U_1^\perp}(v) = (v \cdot v_1)v_1 = \left(\frac{1}{\sqrt{6}}(2,-1,3) \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} =$$

$$= -\frac{3}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$P_{U_1}(v) = v - P_{U_1^\perp}(v)$$

$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \\ 5/2 \end{pmatrix}.$$

(b) $\boxed{a = 1}$

$$v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \longrightarrow v + U_1^\perp = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\rangle.$$

(c) $a \in \mathbb{R}$

$p_{V_a}(v) = 0 \Rightarrow \underbrace{v \in V_a^{\perp}}$

$\begin{cases} v \cdot u_1 = 0 \longrightarrow (2,-1,3) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 5 \neq 0 ! \\ v \cdot u_{2,a} = 0 \end{cases}$

$\Rightarrow \nexists a \in \mathbb{R}$.

(d) $a \in \mathbb{R}$ t.c.

$\| p_{V_a}(v) \| = \sqrt{13}$

$V_a^{\perp} = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle$

$p_{V_a}(v) = v - p_{V_a^{\perp}}(v)$

$V_a^{\perp} : \begin{cases} x + z = 0 \\ (1+a)x - ay + (a-1)z = 0 \end{cases}$

$\begin{cases} z = -x \\ (1+a)x - ay - (a-1)x = 0 \end{cases}$

$\longrightarrow z = -\frac{a}{2} y$

$\begin{cases} z = -x \\ 2x = ay \longrightarrow x = \frac{a}{2} y \end{cases}$

$V_a^{\perp} = \left\langle \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix} \right\rangle$ $\hookrightarrow w_a$

$v_a = \frac{w_a}{\|w_a\|}$

$\|w_a\| = \sqrt{2a^2 + 4}$

$$v_a = \frac{1}{\sqrt{2a^2+4}} \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}$$

$$p_{v_a^\perp}(v) = (v \cdot v_a)\, v_a$$

$$= \frac{1}{(2a^2+4)} \left[(2,-1,3) \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix} \right] \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}$$

$$= \frac{1}{2a^2+4} \big(\underbrace{2a - 2 - 3a}_{-a-2} \big) \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}$$

$$= \frac{-a-2}{2a^2+4} \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}$$

$$p_{v_a}(v) = v - \frac{-a-2}{2a^2+4} \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \frac{a+2}{2a^2+4} \begin{pmatrix} a \\ 2 \\ -a \end{pmatrix}$$

$$= \begin{pmatrix} 2 + \frac{a^2+2a}{2a^2+4} \\ -1 + \frac{a+2}{a^2+2} \\ 3 + \frac{-a^2-2a}{2a^2+4} \end{pmatrix} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} \frac{5a^2+2a+8}{2a^2+4} \\ \frac{-a^2+a+4}{a^2+2} \\ \frac{5a^2-2a+12}{2a^2+4} \end{pmatrix}$$

$$\|p_{U_Q}(v)\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{13}$$

$$1^2 + 2^2 + 3^2 = 13.$$

Ex. 4.

$\mathbb{R}^4$

(2) $v = \begin{pmatrix} 1 \\ 4 \\ -3 \\ -2 \end{pmatrix}$ su V

$$V^\perp = \left\langle \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{= w_1}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{= w_2} \right\rangle \qquad p_V(v) = v - p_{V^\perp}(v)$$

$$v_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_2 = \frac{\overbrace{w_2 - (w_2 \cdot v_1) v_1}^{v_2'}}{\|v_2'\|}$$

$$v_2' = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{3}} \overbrace{(1, 0, 1, 0) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}}^{=1} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ 1 \\ -1/3 \end{pmatrix}$$

$$\|v_2'\|^2 = \frac{4}{9} + \frac{1}{9} + 1 + \frac{1}{9}$$
$$\| = \frac{5}{3}$$

$$v_2 = \sqrt{\frac{3}{5}} \begin{pmatrix} 2/3 \\ -1/3 \\ 1 \\ -1/3 \end{pmatrix}$$

BASE ORTONORMALE DI $V^\perp = \{v_1, v_2\}$

$$p_{V^\perp}(v) = (v \cdot v_1) v_1 + (v \cdot v_2) v_2 =$$

$$= \left((1,4,-3,-2) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} +$$

$$+ \left((1,4,-3,-2) \sqrt{\frac{3}{5}} \begin{pmatrix} 2/3 \\ -1/3 \\ 1 \\ -1/3 \end{pmatrix} \right) \sqrt{\frac{3}{5}} \begin{pmatrix} 2/3 \\ -1/3 \\ 1 \\ -1/3 \end{pmatrix} =$$

$$= \frac{1}{\cancel{3}} \cdot \cancel{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{3}{5} (-3) \begin{pmatrix} 2/3 \\ -1/3 \\ 1 \\ -1/3 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -6/5 \\ 3/5 \\ -9/5 \\ 3/5 \end{pmatrix} = \begin{pmatrix} -1/5 \\ 8/5 \\ -9/5 \\ 8/5 \end{pmatrix}$$

$$p_V(v) = v - p_{V^\perp}(v)$$

(b)

$$V : \begin{cases} x + y + w = 0 \\ x + z = 0 \end{cases} \qquad \begin{cases} w = -x - y \\ z = -x \end{cases}$$

$$V = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

(c) $v = \begin{pmatrix} 1 \\ 4 \\ -3 \\ -2 \end{pmatrix}$ so W

$$W^\perp = \left\langle \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle \quad = w$$

BASE ORTONORM. DI $W^{\perp} = \left\{ \frac{1}{\sqrt{6}} w \right\}$

$$p_{W^{\perp}}(v) = \frac{1}{6}\Big(\underbrace{(1,4,-3,-2)\cdot\begin{pmatrix}2\\0\\1\\1\end{pmatrix}}_{=-3}\Big)\begin{pmatrix}2\\0\\1\\1\end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix}2\\0\\1\\1\end{pmatrix}$$

$$\Rightarrow p_W(v) = v + \frac{1}{2}\begin{pmatrix}2\\0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\4\\-3\\-2\end{pmatrix} + \begin{pmatrix}1\\0\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}2\\4\\-5/2\\-3/2\end{pmatrix}$$

Ex.

$U = \langle u \rangle, \quad u = \begin{pmatrix}1\\-1\\2\\3\end{pmatrix}$

$W = U^{\perp}$

(a) EQ. CARTESIANA + BASE di W?

$w = (x_1, x_2, x_3, x_4) \in W$

$W: u \cdot w = 0$

$$(1,-1,2,3)\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \boxed{x_1 - x_2 + 2x_3 + 3x_4 = 0}$$

$x_2 = x_1 + 2x_3 + 3x_4$

$$\text{BASE di } W = \left\{ \underset{w_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}}, \underset{w_3}{\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}} \right\} = \mathcal{W}$$

(b) $\rightarrow$ GRAM-SCHIMDT

$\mathcal{W} \rightarrow$ BASE ORTOGONALE $= \{ w_1', w_2', w_3' \}$

$$w_1' = w_1$$

$$w_2' = w_2 + \alpha_1 w_1', \quad \alpha_1 = -\frac{w_2 \cdot w_1'}{w_1' \cdot w_1'} = -\frac{2}{2} = -1$$

$$= \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$w_3' = w_3 + \alpha_1 w_1' + \alpha_2 w_2'$$

$$\alpha_1 = -\frac{w_3 \cdot w_1'}{w_1' \cdot w_1'} = -\frac{3}{2} \qquad \alpha_2 = -\frac{w_3 \cdot w_2'}{w_2' \cdot w_2'} = -\frac{3}{3} = -1$$

$$w_3' = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1/2 \\ -1 \\ 1 \end{pmatrix}$$

(c) $v = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix}$

$v_1 \in U$

$v_2 \in U^{\perp} = W$

t.c. $v = \underset{p_U(v)}{v_1} + \underset{p_{U^\perp}(v)}{v_2}$

$\Rightarrow v_2 = v - v_1$

$U = \langle u \rangle \quad u = (1, -1, 2, 3)$

$v_1 = \lambda u \in U, \ \lambda \in \mathbb{R}$

$v_2 = v - \lambda u \in U^{\perp} \iff v_2 \perp u$

$v_2 \cdot u = (v - \lambda u) \cdot u = 0$

$v_2 \cdot u = \underbrace{v \cdot u - \lambda u \cdot u = 0}$

$$\Rightarrow \lambda = \frac{v \cdot u}{u \cdot u} = \frac{(3,-1,5,2)\begin{pmatrix}-1\\2\\3\end{pmatrix}}{(1,-1,2,3)\begin{pmatrix}1\\-1\\2\\3\end{pmatrix}} =$$

$$= \frac{20}{15} = \frac{4}{3}$$

$$v_1 = \frac{4}{3}\begin{pmatrix}1\\-1\\2\\3\end{pmatrix} = \left(\frac{4}{3}, -\frac{4}{3}, \frac{8}{3}, 4\right)$$

$$v_2 = v - v_1 = \begin{pmatrix}3\\-1\\5\\2\end{pmatrix} - \begin{pmatrix}4/3\\-4/3\\8/3\\4\end{pmatrix}$$

$$= \begin{pmatrix}5/3\\1/3\\7/3\\-2\end{pmatrix}$$

(d) $f: \mathbb{R}^4 \to \mathbb{R}^4$

$v \in \mathbb{R}^4 \longmapsto p_U(v)$

$A = M_{\mathcal{E}\to\mathcal{E}}(f) = ?$ $\quad \operatorname{Ker}(f) = ?$

$$p_u(v) = \lambda u = \left(\frac{v \cdot u}{u \cdot u}\right) u$$

$$A = \left(f(e_1) \,\middle|\, f(e_2) \,\middle|\, f(e_3) \,\middle|\, f(e_4) \right)$$

$$f(e_i) = p_u(e_i) = \left(\frac{e_i \cdot u}{u \cdot u}\right) u$$

$$u = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}$$

$$f(e_1) = \left(\frac{e_1 \cdot u}{u \cdot u}\right) u = \frac{1}{15} u = \left(\frac{1}{15}, -\frac{1}{15}, \frac{2}{15}, \frac{1}{5} \right)$$

$$f(e_2) = \left(\frac{e_2 \cdot u}{u \cdot u}\right) u = \left(-\frac{1}{15}, \frac{1}{18}, -\frac{2}{15}, -\frac{1}{5} \right)$$

$$f(e_3) = \left(\frac{e_3 \cdot u}{u \cdot u}\right) u = \left(\frac{2}{15}, -\frac{2}{15}, \frac{4}{15}, \frac{2}{5} \right)$$

$$f(e_4) = \left(\frac{e_4 \cdot u}{u \cdot u}\right) u = \left(\frac{1}{5}, -\frac{1}{5}, \frac{2}{5}, \frac{3}{5}\right)$$

$$A = \begin{pmatrix} \frac{1}{15} & -\frac{1}{15} & \frac{2}{15} & \frac{1}{5} \\ -\frac{1}{15} & \frac{1}{15} & -\frac{2}{15} & -\frac{1}{5} \\ \frac{2}{15} & -\frac{2}{15} & \frac{4}{15} & \frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

$$\Rightarrow \operatorname{Ker} f = U^{\perp}.$$

<u>Ex.</u>

$u_1 = (2, -1, 0, 3)$

$u_2 = (1, 3, -1, 2)$

$f : \mathbb{R}^4 \to \mathbb{R}^2$

$$v \longmapsto f(v) = \begin{pmatrix} u_1 \cdot v \\ u_2 \cdot v \end{pmatrix}, \quad v \in \mathbb{R}^4$$

(2) f LINEARE, $\ker f$, $\operatorname{Im} f$?

$v_1, v_2 \in \mathbb{R}^4$, $\lambda_1, \lambda_2 \in \mathbb{R}$

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \begin{pmatrix} u_1 \cdot (\lambda_1 v_1 + \lambda_2 v_2) \\ u_2 \cdot (\lambda_1 v_1 + \lambda_2 v_2) \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1 u_1 \cdot v_1 + \lambda_2 u_1 \cdot v_2 \\ \lambda_1 u_2 \cdot v_1 + \lambda_2 u_2 \cdot v_2 \end{pmatrix} =$$

$$= \lambda_1 \begin{pmatrix} u_1 \cdot v_1 \\ u_2 \cdot v_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} u_1 \cdot v_2 \\ u_2 \cdot v_2 \end{pmatrix} =$$

$$= \lambda_1 f(v_1) + \lambda_2 f(v_2)$$

$\ker f \to v$ t.c. $f(v) = 0$ $\quad v = (x_1, x_2, x_3, x_4)$

$$f(v) = \begin{pmatrix} u_1 \cdot v \\ u_2 \cdot v \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 + 3x_4 \\ x_1 + 3x_2 - x_3 + 2x_4 \end{pmatrix}$$

$$\ker f : \begin{cases} 2x_1 - x_2 + 3x_4 = 0 \\ x_1 + 3x_2 - x_3 + 2x_4 = 0 \end{cases}$$

$\rightarrow x_2 = 2x_1 + 3x_4$

$\rightarrow x_3 = 7x_1 + 11x_4$

$$\operatorname{Ker} f = \left\langle \begin{pmatrix} 1 \\ 2 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 11 \\ 1 \end{pmatrix} \right\rangle \rightarrow \dim \operatorname{Ker} f = 2$$

$$\Rightarrow \underbrace{\dim \operatorname{Ker} f}_{=2} + \dim \operatorname{Im} f = \dim(\mathbb{R}^4) = 4$$

$\Rightarrow \dim \operatorname{Im} f = 2$

$\Rightarrow \operatorname{Im} f = \mathbb{R}^2$

$$\operatorname{Im} f = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle.$$

(b) $A = M_{\mathcal{E}\to\mathcal{E}}(f) = ?$ $\quad f(v) = \begin{pmatrix} u_1 \cdot v \\ u_2 \cdot v \end{pmatrix}$

$$A = \begin{pmatrix} 2 & -1 & 0 & 3 \\ 1 & 3 & -1 & 2 \end{pmatrix}.$$

$\uparrow f(e_1) \quad \uparrow f(e_2) \quad \uparrow f(e_3) \quad \uparrow f(e_4)$